

Dependent  $T$  and existence of limit models

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## Abstract

Does the class of linear orders have (one of the variants of) the so called  $(\lambda, \kappa)$ -limit model? It is necessarily unique, and naturally assuming some instances of G.C.H. we get some positive results. More generally, letting  $T$  be a complete first order theory and for simplicity assume G.C.H., for regular  $\lambda > \kappa > |T|$  does  $T$  have (variants of) a  $(\lambda, \kappa)$ -limit models, except for stable  $T$ ? For some, yes, the theory of dense linear order, for some, no. Moreover, for independent  $T$  we get negative results. We deal more with linear orders.

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## 0 Introduction

The first part of the introduction is intended for a general mathematical reader. Cantor proved that the structure “the rationals as a linear order” is characterized up to isomorphism by being “a dense linear order with neither first nor last element which is countable”. Hausdorff generalizes this as follows. For transparency assume the G.C.H., the generalized continuum hypothesis, then for every cardinal  $\lambda$  there is a unique linear order  $I$  of cardinality  $\lambda^+$  which is  $\lambda^+$ -dense (i.e. if  $A < C$  are subsets of cardinality  $\leq \lambda$  then for some  $b \in I$  we have  $A < b < C$ ) with neither first nor last elements. This canonical linear order is, in later model theoretic notions, the unique saturated model of the theory  $T_{\text{ord}} = \text{Th}(\mathbb{Q}, <)$  of cardinality  $\lambda^+$  (also the universal homogeneous); note  $T_{\text{ord}}$  is the first order theory of the rational order.

Later Bjarni Jónsson [3], [4] introduced and proved the existence of homogeneous-universal model in cardinality  $\lambda$ , for a quite general class of structures. Morley and Vaught [7] introduced saturated and investigate such models (which are homogeneous universal if we use elementary submodels instead of substructures). Saturated models become a central notion in model theory.

The author in [14] or [8] = [16, Ch.I], introduce abstract elementary classes and there define some variants of  $(\lambda, \kappa)$ -limit models which are again (like the homogeneous universal ones) unique but for the pair of cardinals  $\lambda, \kappa$ ; note that for  $\lambda = \kappa = \mu^+$  this is the previous case. So natural questions are: what about elementary classes, i.e. first order theories? and what about the class of linear orders?

By [12] if  $T$  is low enough (so called stable) there are existence theorems but, e.g. the theory of linear order is not stable.

What are our main results? First, a result meaningful also to one with very little set theoretic background. If  $\lambda = \lambda^{<\lambda}$ , e.g.  $\lambda = \mu^+ = 2^\mu$ , then in addition to the unique (up to isomorphisms) linear order which Hausdorff discovers, for  $\kappa = \aleph_0$  or just  $\kappa \leq \lambda$  which is a successor (or just a so called regular) there is a  $(\lambda, \kappa)$ -limit linear order and it is unique up to isomorphisms. We can also

have a characterization (as in the case of Hausdorff), though not so elegant; see §1 which do not require model theoretic background, see Theorem 1.1. There are stronger versions of “ $(\lambda, \kappa)$ -limit models” ( $(\lambda, \kappa)$ -superlimit) for which we show non-existence, see §3.

Second, in model theoretic terms this shows that having  $(\lambda, \kappa)$ -limit model is satisfied by some (complete first order) theories  $T$  which are not stable; all this in §1. So does every  $T$  have such models? In §2 comes the other major result of this work: the answer in general, no, e.g. for (the first order theory) Peano arithmetic, see Theorem 2.3. Moreover, there is a reasonable natural sufficient condition: the theory  $T$  is so called dependent, this is Theorem 2.9.

Those complementary results lead to the main conjectures arising from this work on existence of  $(\lambda, \kappa)$ -limit models and to the generic pair conjecture. They essentially say that the above mentioned sufficient condition, “ $T$  is dependent” is the right one, each dealing with a variant of the question (the first: any relevant  $\kappa$ , the second: the parallel for  $\kappa = 2$ ).

The question can be rephrased (under G.C.H., restricting ourselves to successor cardinality  $\aleph_{\varepsilon+1}$ ) as follows: assume  $\langle M_\alpha : \alpha < \aleph_{\varepsilon+2} \rangle$  is a  $\prec$ -increasing continuous sequence of models of the first order complete  $T$ ,  $\|M_\alpha\| = \aleph_{\varepsilon+1}$  and  $M = \bigcup \{M_\alpha : \alpha < \aleph_{\varepsilon+2}\}$  is saturated (e.g. Hausdorff linear order of cardinality  $\aleph_{\varepsilon+2}$ ,  $T = T_{\text{ord}}$ ). Let  $\mathbf{n}_{\aleph_{\varepsilon+1}}(T) = \text{Min}\{|\{M_\alpha / \cong : \alpha \in E\}| : E \text{ a closed unbounded subset of } \aleph_{\varepsilon+2}\}$ . Now the existence of  $(\aleph_{\varepsilon+1}, \kappa)$ -limit model for every regular  $\kappa < \aleph_{\varepsilon+2}$  implies  $\mathbf{n}_T(\aleph_{\varepsilon+1}) = |\varepsilon + 2|$ , in fact for some such  $E$  for any  $\kappa$  all the models  $\{M_\delta : \delta \in E\}$  has cofinality  $\kappa$  are pairwise isomorphic. Our non-existence results give  $\mathbf{n}_{\aleph_{\varepsilon+1}}(T) = \aleph_{\varepsilon+2}$ .

The rest of the introduction we assume more background.

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We continue [12] and [17].

The problem in [12] is when does (a first order theory)  $T$  have a model  $M$  of cardinality  $\lambda$  which is (one of the variants of) a limit model for cofinality  $\kappa$ , in the cases not covered by [12, 0.8] (or [14, 3.3,3.2], [8, 3.6,3.5]). More accurately, there are some versions of limit models, “ $M$  is a  $(\lambda, \kappa)$ - $x$ -limit model of  $T$ ” mainly “ $(\lambda, \kappa)$ -i.md. limit”, see Definition 0.8; (though we deal with others, too) the most natural case to try is  $\lambda = \lambda^{<\lambda} > \kappa = \text{cf}(\kappa) > |T|$ .

Note that if  $T$  has (any version of) a limit model of cardinality  $\lambda$  then there is a universal  $M \in \text{Mod}_\lambda(T)$ . Now we know that if  $\lambda = 2^{<\lambda} > |T|$  then there is a universal  $M \in \text{Mod}_\lambda(T)$  (see e.g. [2]). But for other cardinals it is “hard to have a universal model”, see history [5] and [6]. E.g. if  $T$  has the strict order property, then, by Kojman-Shelah [5] there are ZFC non-existence results (a major case, for regular  $\lambda$  is when  $(\exists \mu)(\mu^+ < \lambda \wedge 2^\mu > \lambda)$ ). In at least one case,  $\lambda = \aleph_1 < 2^{\aleph_0}$  consistently we do not have a universal model, see [13].

Stable theories have limit models (in many cases); hence it is natural to ask:

Question 1: Assume  $\lambda = \lambda^{<\kappa} > \kappa > |T|$ . Does the existence of a  $(\lambda, \kappa)$ -md.-limit model of  $T$  imply  $T$  is stable?

This is quite reasonable but in Theorem 1.1 we find a counterexample, in fact, one everyone knows about: the theory  $T_{\text{ord}}$  of dense linear orders (see 0.12). This per se is a continuation of Hausdorff result, revealing some canonical linear ordres. Returning to the family of elementary classes, i.e. first order theories, it is natural to ask:

Question 2: Does  $T$  have a  $(\lambda, \kappa)$ -i.md.-limit model whenever  $\lambda = \lambda^{<\lambda} > \kappa + |T|$  for every unstable  $T$ ?

For non-existence results it is natural to look at  $T$  dissimilar to  $T_{\text{ord}}$ . As  $T_{\text{ord}}$  is prototypical of dependent theories, it is natural to look for independent theories. A strong, explicit version of  $T$  being independent is having the strong independence property (see Definition 2.4), e.g. Peano arithmetic has. We prove that for such  $T$  there are no limit models (2.3). But the strong independence property does not seem a good dividing line. The independence property is a good candidate for being a meaningful dividing line.

Question 3: If  $T$  is independent, does  $T$  have a  $(\lambda, \kappa)$ -i.md.-limit model (with  $\lambda = \lambda^{<\lambda} > \kappa > |T|$ )?

We work harder (than in 2.3) to prove (in 2.9) the negative answer for every independent  $T$  (for many cardinals), i.e. with the independence property though a weaker version meaning we prove non-existence of a stronger version of “ $(\lambda, \kappa)$ -limit model”.

This makes us

**Conjecture 0.1.** Any dependent  $T$  has  $(\lambda, \kappa)$ -i.md.-limit model.

Toward this end we intend to continue the investigation of types for dependent  $T$ .

We shall also consider a property  $\text{Pr}_{\lambda, \kappa}(T)$  (and the stronger  $\text{Pr}_{\lambda, \kappa}^2(T)$ ), see Definition 2.5, which are relatives of “there is no  $(\lambda, \kappa)$ - $x$ -limit model”; i.e. non-existence results for independent  $T$  holds for  $\lambda = \lambda^{<\lambda} \geq \kappa = \text{cf}(\kappa), \lambda > |T|$ . For  $\lambda > \kappa$  this strengthens “there is no  $(\lambda, \kappa)$ -i.md.-limit model”. But  $\lambda = \kappa$  is a new non-trivial case and it is also a candidate to be “an outside equivalent condition for  $T$  being dependent”.

The most promising among the relatives (for having a dichotomy) is the following conjecture (the assumption  $2^\lambda = \lambda^+$  is just for simplicity).

**Conjecture 0.2.** The generic pair conjecture

Assume  $\lambda = \lambda^{<\lambda} > |T|$  and  $2^\lambda = \lambda^+$  (for transparency) and  $M_\alpha \in \text{EC}_\lambda(T)$  is  $<$ -increasing continuous for  $\alpha < \lambda^+$  with  $\bigcup\{M_\alpha : \alpha < \lambda^+\} \in \text{EC}_{\lambda^+}(T)$  saturated. Then  $T$  is dependent iff for some club  $E$  of  $\lambda^+$  for all pairs  $\alpha < \beta < \lambda^+$  from  $E$  both of cofinality  $\lambda$ ,  $(M_\beta, M_\alpha)$  has the same isomorphism type (we denote this property of  $T$  by  $\text{Pr}_\lambda^2(T)$ ), see Definition 2.5).

Here we prove that for independent  $T$ , a strong version of the conjecture holds.

In §2, we also prove the parallel of what we say above. In §3 we prove that  $(\lambda, \kappa)$ -superlimit models does not exist even for  $T = T_{\text{ord}}$ . This work is continued in [18], [10], [9], [11] and Kaplan-Lavi-Shelah [19].

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Now we define some versions of “ $M$  is a  $(\lambda, S)$ - $x$ -limit model” and for them “ $\bar{M}$  obeys a  $(\lambda, T)$ - $x$ -function”.

**Notation 0.3.** 1) Let  $T$  denote a complete first order theory.

2) Let  $\tau_T = \tau(T), \tau_M = \tau(M)$  be the vocabulary of  $T, M$  respectively.

**Definition 0.4.** 1) For any  $T$  let  $\text{EC}(T) = \{M : M \text{ a } \tau_T\text{-model of } T\}$ .

2)  $\text{EC}_\lambda(T) = \{M \in \text{EC}(T) : M \text{ is of cardinality } \lambda\}$  and  $\text{EC}_{\lambda, \kappa}(T) = \{M \in \text{EC}_\lambda(T) : M \text{ is } \kappa\text{-saturated}\}$ .

3) We say  $M \in \text{EC}(T)$  is  $\lambda$ -universal when every  $N \in \text{EC}_\lambda(T)$  can be elementarily embedded into  $M$ .

- 4) We say  $M \in \text{EC}(T)$  is universal when it is  $\lambda$ -universal for  $\lambda = \|M\|$ .  
 5) For  $T \subseteq T'$  let

$$\text{PC}(T', T) = \{M \upharpoonright \tau_T : M \text{ is model of } T'\}$$

$$\text{PC}_\lambda(T', T) = \{M \in \text{PC}(T', T) : M \text{ is of cardinality } \lambda\}.$$

**Definition 0.5.** Given  $T$  and  $M \in \text{EC}_\lambda(T)$  we say that  $M$  is a  $(\lambda, \kappa)$ -superlimit model when:  $M$  is a  $\lambda$ -universal model of cardinality  $\lambda$  and if  $\delta < \lambda^+$  is a limit ordinal such that  $\text{cf}(\delta) = \kappa$ ,  $\langle M_\alpha : \alpha \leq \delta \rangle$  is  $\prec$ -increasing continuous, and  $M_{\alpha+1}$  is isomorphic to  $M$  for every  $\alpha < \delta$  then  $M_\delta$  is isomorphic to  $M$ .

**Remark 0.6.** We shall use:

- (a)  $(\lambda, \kappa)$ -i.md.-limit in 1.1, (existence for  $T_{\text{ord}}$ )
- (b)  $(\lambda, \kappa)$ -wk-limit in 2.3, (non-existence from “ $T$  is strongly independent”)
- (c)  $(\lambda, \kappa)$ -md.-limit in 2.9, (non-existence for independent  $T$ )
- (d)  $(\lambda, \kappa)$ -i.st.-limit for  $T_{\text{ord}}$ : 3.12 and 3.5(3), 3.7(3), (on characterization) for  $T_{\text{ord}}$ )
- (e)  $(\lambda, \kappa)$ -superlimit in 3.10 (non-existence).

Recall the definition of some versions of “ $(\lambda, \kappa)$ -limit model”.

**Convention 0.7.** In this work let “ $M$  is  $(\lambda, S)$ -limit” mean “ $M$  is  $(\lambda, S)$  – md-limit, see Definition below; similarly for  $(\lambda, \kappa)$ .”

**Definition 0.8.** Let  $\lambda$  be a cardinal  $\geq |T|$ . For parts 3) - 5) but not 6), for simplifying the presentation we assume the axiom of global choice; alternatively restrict yourself to models with universe an ordinal  $\in [\lambda, \lambda^+)$ . Below if  $S = \{\delta < \lambda^+ : \text{cf}(\delta) = \kappa\}$  then instead  $(\lambda, S)$  we may write  $(\lambda, \kappa)$ , this is the main case.

1) Let  $S \subseteq \lambda^+$  be stationary. A model  $M \in \text{EC}_\lambda(T)$  is called  $(\lambda, S)$ -st-limit (or  $S$ -strongly limit or  $(\lambda, S)$ -strongly limit) when for some function:  $\mathbf{F} : \text{EC}_\lambda(T) \rightarrow \text{EC}_\lambda(T)$  we have:

- (a) for  $N \in \text{EC}_\lambda(T)$  we have  $N \prec \mathbf{F}(N)$
- (b) if  $\delta \in S$  is a limit ordinal and  $\langle M_i : i < \delta \rangle$  is a  $\prec$ -increasing continuous sequence <sup>1</sup> in  $\text{EC}_\lambda(T)$  obeying  $\mathbf{F}$  which means  $i < \delta \Rightarrow \mathbf{F}(M_{i+1}) \prec M_{i+2}$ , then  $M \cong \cup\{M_i : i < \delta\}$ .

2) Let  $S \subseteq \lambda^+$  be stationary.  $M \in \text{EC}_\lambda(T)$  is called  $(\lambda, S)$ -nr-limit (or  $S$ -normally limit, or may omit nr/normally) when for some function  $\mathbf{F} : \text{EC}_\lambda(T) \rightarrow \text{EC}_\lambda(T)$  we have:

- (a) for every  $N \in \text{EC}_\lambda(T)$  we have  $N \prec \mathbf{F}(N)$

<sup>1</sup>No loss if we add  $M_{i+1} \cong M$ , so this simplifies the demand on  $\mathbf{F}$ , i.e., only  $\mathbf{F}(M)$  is required

(b) if  $\langle M_i : i < \lambda^+ \rangle$  is a  $\prec$ -increasing continuous sequence of members of  $\text{EC}_\lambda(T)$ ,  $\mathbf{F}(M_{i+1}) \prec M_{i+2}$  then for some closed unbounded <sup>2</sup> subset  $C$  of  $\lambda^+$ ,

$$[\delta \in S \cap C \Rightarrow M_\delta \cong M].$$

2A)  $M \in \text{EC}_\lambda(T)$  is  $(\lambda, S)$ -limit<sup>+</sup> when if  $\langle M_\alpha : \alpha < \lambda^+ \rangle$  is  $\subseteq$ -increasing and continuous and  $\alpha < \lambda^+ \Rightarrow M_{\alpha+1} \cong M$  then for some club  $E$  of  $\lambda$  we have  $\alpha \in E \cap S \rightarrow M_\alpha \cong M$ . Notice that being a  $(\lambda, S)$ -limit<sup>+</sup> implies being a  $(\lambda, S)$ -nr-limit.

3) We define “ $M$  is  $(\lambda, S)$ -wk-limit”, “ $(\lambda, S)$ -md-limit” like “ $(\lambda, S)$ -nr-limit”, “ $(\lambda, S)$ -st-limit” respectively by demanding that the domain of  $\mathbf{F}$  is the family of  $\prec$ -increasing continuous sequences of members of  $\text{EC}_\lambda(T)$  of length  $< \lambda^+$  and replacing “ $\mathbf{F}(M_{i+1}) \prec M_{i+2}$ ” by “ $M_{i+1} \prec \mathbf{F}(\langle M_j : j \leq i+1 \rangle) \prec M_{i+2}$ ”. (They are also called  $S$ -weakly limit,  $S$ -medium limit, respectively.)

3A) We replace “limit” by “limit<sup>-</sup>” if “ $\mathbf{F}(M_{i+1}) \prec M_{i+2}$ ”, “ $M_{i+1} \prec \mathbf{F}(\langle M_j : j \leq i+1 \rangle) \prec M_{i+2}$ ” are replaced by “ $\mathbf{F}(M_i) \prec M_{i+1}$ ”, “ $M_i \prec \mathbf{F}(\langle M_j : j \leq i \rangle) \prec M_{i+1}$ ” respectively.

4) If  $S = \lambda^+$  then we may omit  $S$  (in parts (3), (4), (5)).

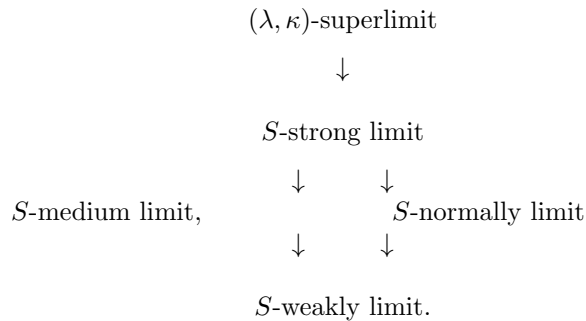
5) For  $\Theta \subseteq \{\mu : \mu \leq \lambda \text{ and } \mu \text{ is regular}\}$ ,  $M$  is  $(\lambda, \Theta)$ -strongly limit if  $M$  is  $\{\delta < \lambda^+ : \text{cf}(\delta) \in \Theta\}$ -strongly limit in the sense of 1). Similarly for the other notions (where  $\Theta \subseteq \{\mu : \mu \text{ regular } \leq \lambda\}$  is non-empty and  $S_1 \subseteq \{\delta < \lambda^+ : \text{cf}(\delta) \in \Theta\}$  is a stationary subset of  $\lambda^+$ ). If we do not write  $\lambda$  we mean  $\lambda = \|M\|$ .

6) We say that  $M \in K_\lambda$  is  $(\lambda, S)$ -i.st-limit (or  $S$ -invariantly strong limit) when in part (3),  $\mathbf{F}$  is just a subset of  $\{(M, N)/\cong : M \prec N \text{ are from } \text{EC}_\lambda(T)\}$  and in clause (b) of part (3) we replace “ $\mathbf{F}(M_{i+1}) \prec M_{i+2}$ ” by “ $(\exists N)(M_{i+1} \prec N \prec M_{i+2} \wedge ((M_{i+1}, N)/\cong) \in \mathbf{F})$ ”. But abusing notation we still write  $N = \mathbf{F}(M)$  instead  $((M, N)/\cong) \in \mathbf{F}$ . Similarly with the other notions, i.e., we use the isomorphism type of  $\bar{M} \hat{\ } \langle N \rangle$ .

**Observation 0.9.** 1) If  $\mathbf{F}_1, \mathbf{F}_2$  are as above and  $\mathbf{F}_1(N) \prec \mathbf{F}_2(N)$  (or  $\mathbf{F}_1(\bar{M}) \prec \mathbf{F}_2(\bar{M})$ ) whenever defined then if  $\mathbf{F}_1$  is a witness so is  $\mathbf{F}_2$ .

2) All versions of limit models imply being a universal model in  $\text{EC}_\lambda(T)$ .

3) Obvious implication diagram: For stationary  $S \subseteq S_\kappa^{\lambda^+}$  as in 0.8(7):




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<sup>2</sup>We can use a filter as a parameter

**Claim 0.10.** Assume  $\lambda = \lambda^{<\kappa} \geq |T|$  and  $\kappa$  is regular and  $M$  is a model of  $T$  of cardinality  $\lambda$ . Then the following conditions are equivalent (assume the universal axiom of choice or restrict ourselves below to models with universe  $\subseteq \lambda^+$ ):

- (a)  $M$  is  $(\lambda, \kappa)$ -md-limit
- (b) in the following game the isomorphism player has a winning strategy. A play last  $\kappa$ -moves, in the  $i$ -th move the anti-isomorphism player chooses  $M_\alpha \in \text{EC}_\lambda(T)$  such that  $\langle M_\beta : \beta \leq \alpha \rangle$  is  $\prec$ -increasing continuous and  $\alpha = \beta + 1 \Rightarrow M'_\beta \prec M_\alpha$  and the isomorphism player chooses  $M'_\alpha$  such that  $M_\alpha \prec M'_\alpha \in \text{EC}_\lambda(T)$ . The isomorphism player wins a play when  $\bigcup \{M_\alpha : \alpha < \kappa\}$  is isomorphic to  $M$
- (c) there is a function  $\mathbf{F}$  with domain  $\{\bar{M} : \bar{M} \text{ a } \prec\text{-increasing continuous sequence of members of } \text{EC}_\lambda(T) \text{ of length } < \kappa\}$  such that  $i < \text{lg}(\bar{M}) \Rightarrow M_i \prec \mathbf{F}(\bar{M})$  and if  $\bar{M} = \langle M_i : i \leq \kappa \rangle$  is  $\prec$ -increasing continuous sequence of members of  $\text{EC}_\lambda(T)$  and  $i < \kappa \Rightarrow \mathbf{F}(\bar{M} \upharpoonright (2i+2)) \prec M_{2i+2}$  then  $M_\kappa \cong M$
- (d) there is a function  $\mathbf{F}$  such that: if  $\langle M_i : i \leq \kappa \rangle$  is  $\prec$ -increasing continuous in  $\text{EC}_\lambda(T)$  and for some sequence  $\langle M'_i : i < \kappa \rangle$  we have  $M_i \prec M'_i \in \text{EC}_\lambda(T)$  and  $i < \kappa \Rightarrow M'_i \prec \mathbf{F}(\langle M_j : j \leq i \rangle \wedge \langle M'_j \rangle) = M_{i+1}$  (we say  $\bar{M}$  obeys  $\mathbf{F}$ ) then  $\bigcup \{M_i : i < \kappa\} \cong M$
- (e) in  $\mathbf{V}^{\text{Levy}(\lambda^+, 2^\lambda)}$  we have: if  $\langle M_\alpha : \alpha < \lambda^+ \rangle$  is  $\prec$ -increasing continuous,  $M_\alpha \in \text{EC}_\lambda(T)$ , and  $\bigcup \{M_\alpha : \alpha < \lambda^+\} \in \text{EC}_{\lambda, \lambda}(T)$  then for some club  $E$  of  $\lambda^+$  we have  $\delta \in E \wedge \text{cf}(\delta) = \kappa \Rightarrow M_\delta \cong M$
- (f) like (e) for any  $\lambda^+$ -complete forcing notion  $\mathbb{P}$  such that  $\Vdash_{\mathbb{P}} "2^\lambda = \lambda^+"$ .

*Proof.* As  $(\text{EC}_\lambda(T), \prec)$  has the JEP (joint embedding property) and the amalgamation property this is straightforward.

E.G.

(f)  $\Rightarrow$  (b):

Let  $\langle \underline{M}_\alpha : \alpha < \lambda^+ \rangle$  be a  $\mathbb{P}$ -name of a  $\prec$ -increasing continuous sequence of members of  $\text{EC}_\lambda(T)$  with union in  $\text{EC}_{\lambda, \lambda}(T)$  and  $\underline{E}$  a  $\mathbb{P}$ -name of a club of  $\lambda^+$  such that  $\delta \in \underline{E} \wedge \text{cf}(\delta) = \kappa \Rightarrow M_\delta \cong M$ ; clearly it exists by clause (f) which we are assuming. We now define a strategy  $\mathbf{st}$  for the isomorphic player: together with choosing  $M'_\alpha$  the isomorphic player chooses  $(\gamma_\alpha, p_\alpha, h_\alpha)$  such that

- <sub>1</sub>  $M_\alpha \prec M'_\alpha \in \text{EC}_\lambda(T)$  as demanded in (b)
- <sub>2</sub>  $p_\alpha \in \mathbb{P}$  and  $\beta < \alpha \Rightarrow \mathbb{P} \Vdash "p_\beta \leq p_\alpha"$
- <sub>3</sub>  $\gamma_\alpha < \lambda^+$  and  $\beta < \alpha \Rightarrow \gamma_\beta < \gamma_\alpha$
- <sub>4</sub>  $p_\alpha \Vdash_{\mathbb{P}} "h_\alpha \text{ is an isomorphism from } M'_\alpha \text{ onto } \underline{M}_{\gamma_\alpha}"$
- <sub>5</sub> if  $\beta < \alpha$  then  $h_\beta \subseteq h_\alpha$ .

□<sub>0.10</sub>

Like 0.10 but for the invariant version we note

**Claim 0.11.** For  $M \in \text{EC}_\lambda(T)$  the following are equivalent (and seemingly stronger than the conditions in 0.10):

(a)'  $M$  is  $(\lambda, \kappa)$ -i.md-limit (that is invariantly medium  $(\lambda, \kappa)$ -limit)

(d)' there is a class  $\mathbf{F}$  such that:

( $\alpha$ )  $\mathbf{F} \subseteq \{\bar{M} : \bar{M} = \langle M_i : i \leq \alpha \rangle \text{ for some } \alpha \leq \kappa \text{ is } \prec\text{-increasing continuous, } \{M_i : i \leq \alpha\} \subseteq \text{EC}_\lambda(T)\}$  and  $\mathbf{F}$  is closed under isomorphisms

( $\beta$ ) if  $\bar{M} = \langle M_i : i \leq \alpha \rangle \in \mathbf{F}$  and  $M_\alpha \prec M'_\alpha \in \text{EC}_\lambda(T)$  then for some  $M_{\alpha+1}$  we have  $M'_\alpha \prec M_{\alpha+1}$  and  $\bar{M} \hat{\ } \langle M_{\alpha+1} \rangle \in \mathbf{F}$

( $\gamma$ ) for  $\alpha$  limit  $\langle M_i : i \leq \alpha \rangle \in \mathbf{F}$  iff  $j < \alpha \Rightarrow \langle M_i : i \leq j \rangle \in \mathbf{F}$  and  $M_\alpha = \cup \{M_i : i < \alpha\}$

( $\delta$ ) if  $\langle M_i : i \leq \kappa \rangle \in \mathbf{F}$  then  $M_\kappa \cong M$

(d)'' there is  $\mathbf{F}$  such that:

( $\alpha$ )  $\mathbf{F}$  is a function with domain  $\{\bar{M} : \bar{M} = \langle M_i : i \leq \alpha + 1 \rangle \text{ for some } \alpha < \kappa \text{ is } \prec\text{-increasing continuous in } \text{EC}_\lambda(T)\}$  such that  $M_{\alpha+1} \prec \mathbf{F}(\langle M_i : i \leq \alpha + 1 \rangle)$  but  $(\bar{M} \hat{\ } \mathbf{F}(M))$  is unique only up to isomorphism

( $\beta$ ), ( $\gamma$ ), ( $\delta$ ) as in (d)'.

**Definition 0.12.** 1)  $T_{\text{ord}}$  is the theory of dense linear order with neither first nor last element.  
 2)  $T_{\text{rd}}$  is the theory of linear orders, (recall that  $T_{\text{ord}}$  the theory of dense linear order with neither first nor last element).

**Definition 0.13.** 1) We say that  $(C_1, C_2)$  is a cut of  $M \in \text{EC}(T_{\text{rd}})$  when:

(a)  $C_1$  is an initial segment of  $M$

(b)  $C_2$  is an end-segment of  $M$

(c)  $C_1 \cap C_2 = \emptyset$

(d)  $C_1 \cup C_2 = M$ .

2) For a cut  $(C_1, C_2)$  of  $M$ , let  $\text{cf}(C_1, C_2)$ , the cofinality of the cut  $(C_1, C_2)$ , be the pair  $(\theta_1, \theta_2)$  when

(a)  $\theta_1$  is the cofinality of  $C_1$ , i.e. of  $M \upharpoonright C_1$  (can be 0, 1 or a regular cardinal  $\in [\aleph_0, \lambda)$ )

(b)  $\theta_2$  is the cofinality of  $C_2$  inverted.

**Definition 0.14.**  $\dot{I}(\lambda, T)$  is the number of  $M \in \text{EC}_\lambda(T)$  up to isomorphism.

**Definition 0.15.** 1) Fixing  $T, \varphi(\bar{x}, \bar{y})$  is an independent formula when for every  $n$  and  $M \models T$  for some  $\bar{a}_\ell \in {}^{\ell}g(\bar{y})M$  for  $\ell < n$ , for every  $u \subseteq n, M \models (\exists \bar{x}) \bigwedge_{\ell < n} \varphi(\bar{x}, \bar{a}_\ell)^{\text{if}(\ell \in u)}$ .

2)  $T$  is independent iff some  $\varphi(x, \bar{y})$  is independent.

**Notation 0.16.**  $\varphi^{\text{if}(\mathbf{t})}$  is  $\varphi$  if  $\mathbf{t}$  is true or 1,  $\neg\varphi$  if  $\mathbf{t}$  is false or 0.

**Definition 0.17.** 1)  $\lambda^{<\kappa>\text{tr}}$  is  $\sup\{|\mathcal{T} \cap {}^\kappa\lambda| : \mathcal{T} \subseteq {}^\kappa\lambda \text{ is closed under initial segments and } \varepsilon < \kappa \Rightarrow |\mathcal{T} \cap {}^\varepsilon\lambda| \leq \lambda\}$ .

2)  $\mathbf{U}_\kappa(\lambda) = \min\{|\mathcal{F}| : \mathcal{F} \text{ is a set of functions from } \kappa \text{ to } \lambda \text{ such that } f \neq g \in \mathcal{F} \Rightarrow \kappa > \{i < \kappa : f(i) = g(i)\}\}$ .

## 1 Dense linear order has medium limit models

**Theorem 1.1.** Assume  $\lambda = \lambda^{<\lambda} > \kappa = \text{cf}(\kappa)$ . Then  $T_{\text{ord}}$  has an invariantly medium  $(\lambda, \kappa)$ -limit model.

**Remark 1.2.** 1) We use condition (d)' from 0.11, we may use it as a definition.

2) So a model of  $T_{\text{ord}}$  is a dense linear order with neither first nor last element and  $\prec$  for models of  $T_{\text{ord}}$  is just  $\subseteq$  and saturated means  $\lambda$ -dense for models of  $T_{\text{ord}}$  of cardinality  $\lambda$ .

3) We actually prove a result with  $\mathbf{F}$  of a simple kind, dealing with  $\mathbf{F}$  acting on pairs of models,  $\cup\{M_i : i < \kappa\}$  is isomorphic to the  $(\lambda, \kappa)$ -i.md.-limit model when  $\langle M_i : i < \kappa \rangle$  is  $\prec$ -increasing continuous sequence of linear orders such that for any  $i_1 < i_2 < \kappa$  for some  $i_3 \in (i_2, \kappa)$  we have  $\mathbf{F}(M_{i_1}, M_{i_3}) \prec M_{i_3+1}$ .

4) On cuts and their cofinalities 0.13.

**Remark 1.3.** Concerning  $\otimes_{\bar{M}}^\kappa$  in the beginning of the proof of 1.1.

1) It is a characterization of the invariantly medium  $(\lambda, \kappa)$ -model. We shall return to this in §3.

2) Clause (f) almost implies clause (d).

Clause (f) implies (h)<sub>1</sub>; why? use  $A = \emptyset, B = M_i$ . Also clause (f) implies (h)<sub>2</sub>; why? use  $A = M_i, B = \emptyset$ .

Lastly, clause (f) implies (i)<sub>1</sub>; why? use  $A' = A, B' = \{c \in M_i : A < c\}$  and clause (f) implies (i)<sub>2</sub> similarly.

2) Note that clause (f) is equivalent to (i)<sub>1</sub> + (i)<sub>2</sub>.

*Proof.* First we say that  $\bar{M}$  is a fast  $(\lambda, \kappa)$ -sequence (of models of  $T_{\text{ord}}$ ) when:

$$\otimes_{\bar{M}}^\kappa \quad (a) \quad \bar{M} = \langle M_i : i \leq \kappa \rangle$$

(b)  $M_i$  is  $\prec$ -increasing continuous

(c)  $M_i$  is a model of  $T_{\text{ord}}$  of cardinality  $\lambda$

(d)  $M_i$  is saturated if  $i$  is a non-limit ordinal



- (e) if  $i < \kappa$  and  $a \in M_{i+1} \setminus M_i$  then  $M_{i+1} \upharpoonright \{b \in M_{i+1} \setminus M_i : (\forall c \in M_i) [(c < b) \equiv (c < a)]\}$  is a saturated model of  $T_{\text{ord}}$  of cardinality  $\lambda$
- (f) if  $i < \kappa$  and  $A, B \subseteq M_i$  and  $A < B$  (i.e.  $(\forall a \in A)(\forall b \in B)(a <_{M_i} b)$ ) and  $A$  or  $B$  has cardinality  $< \lambda$ , then for some  $c \in M_{i+1} \setminus M_i$  we have  $A < c < B$ ; this includes  $A, B$  singletons but it is enough to have this when  $c \in M_i \Rightarrow \neg(A < c < B)$ ; note that we say “A or B . . .”
- (g)<sub>1</sub> if  $i < j < \kappa$  and  $a \in M_j \setminus M_i$ , then for some  $d \in M_{j+1} \setminus M_j$  we have
- (α) if  $b \in M_i$  and  $b <_{M_j} a$  then  $b <_{M_{j+1}} d$
- (β) if  $c \in M_j$  and  $(\forall b \in M_i)(b <_{M_j} a \Rightarrow b <_{M_j} c)$  then  $d <_{M_{j+1}} c$
- (g)<sub>2</sub> if  $i < j < \kappa$  and  $a \in M_j \setminus M_i$  then for some  $d \in M_{j+1} \setminus M_j$  we have
- (α) if  $b \in M_i$  and  $a <_{M_j} b$  then  $d <_{M_{j+1}} b$
- (β) if  $c \in M_j$  and  $(\forall b \in M_i)(a <_{M_j} b \Rightarrow c <_{M_j} b)$  then  $c <_{M_{j+1}} d$
- (h)<sub>1</sub> for  $i < \kappa$  there is  $b \in M_{i+1} \setminus M_i$  such that  $a \in M_i \Rightarrow a <_{M_{i+1}} b$
- (h)<sub>2</sub> for  $i < \kappa$  there is  $b \in M_{i+1} \setminus M_i$  such that  $a \in M_i \Rightarrow b <_{M_{i+1}} a$
- (i)<sub>1</sub> if  $A \subseteq M_i, i < \kappa$  and  $|A| < \lambda$  then for some  $c \in M_{i+1} \setminus M_i$  we have  $(\forall d \in M_i)(d <_{M_{i+1}} c \leftrightarrow (\exists a \in A)(d \leq_{M_i} a))$
- (i)<sub>2</sub> if  $A \subseteq M_i, i < \kappa$  and  $|A| < \lambda$  then for some  $c \in M_{i+1} \setminus M_i$  we have  $(\forall d \in M_i)(c <_{M_{i+1}} d \leftrightarrow (\forall a \in A)(a \leq_{M_i} d))$
- (j) if  $i < \kappa$  and  $a <_{M_i} b$  then for some  $c \in (a, b)_{M_{i+1}} \setminus M_i$  the orders  $M_i \upharpoonright \{d \in M_i : d <_{M_{i+1}} c\}$  and the inverse of  $M_i \upharpoonright \{d \in M_i : c <_{M_{i+1}} d\}$  have cofinality  $\lambda$ .

Clearly:

☒ it is enough to prove ☒<sub>1</sub> + ☒<sub>2</sub> where

☒<sub>1</sub> there is  $\mathbf{F}$  such that

(α)  $\text{Dom}(\mathbf{F}) = \{\bar{M} : \bar{M} = \langle M_i : i < \alpha + 2 \rangle$  is  $\prec$ -increasing, continuous  
 $M_i \in \text{EC}_\lambda(T_{\text{ord}})$  and  $\alpha < \lambda^+\}$

(β)  $M_{\alpha+1} \prec \mathbf{F}(\langle M_i : i < \alpha + 2 \rangle) \in \text{EC}_\lambda(T_{\text{ord}})$

( $\gamma$ )  $\mathbf{F}$  is invariant, i.e. if  $\bar{M}_1 \cong \bar{M}_2$  then  $(\bar{M}_1, \mathbf{F}(\bar{M}_1)) \cong (M_2, \mathbf{F}(\bar{M}_2))$

( $\delta$ ) if  $\bar{M} = \langle M_i : i \leq \kappa \rangle$  is an  $\prec$ -increasing continuous sequence of members of  $\text{EC}_\lambda(T_{\text{ord}})$  obeying  $\mathbf{F}$  in 0.11(d)'' sense then  $\otimes_{\bar{M}}^\kappa$

$\boxtimes_2$  if  $\otimes_{\bar{M}_1}^\kappa$  and  $\otimes_{\bar{M}_2}^\kappa$  then  $M_\kappa^1 \cong M_\kappa^2$ .

Why is clause  $\boxtimes_1$  true?:

How do we choose  $\mathbf{F}$ ?

Reading the definition of  $\otimes_{\bar{M}}^\kappa$  this should be clear: all our demands on  $M_{j+1}$  when we are given  $\langle M_i : i \leq j \rangle$  and  $M'_j$  can be fulfilled. We first choose  $\mathcal{P}_{\langle M_i : i \leq j \rangle} = \{(A, B) : (A, B) \text{ a cut of } M_j \text{ such that } A \text{ has cofinality } < \lambda \text{ or the inverse of } B \text{ has cofinality } < \lambda \text{ or for some } i < j \text{ and } a \in M_j \setminus M_i \text{ the set } \{b \in M_i : b <_{M_j} a\} \text{ is unbounded in } A \text{ or for some } i < j \text{ and } a \in M_j \setminus M_i \text{ the set } \{b \in M_i : a <_{M_j} b\} \text{ is unbounded from below in the set } B\}$ .

Second, choose  $M_{j+1} = \mathbf{F}(\langle M_i : i \leq j \rangle \wedge \langle M'_j \rangle)$  such that  $M'_j \prec M_{j+1}$  and any cut  $(A, B) \in \mathcal{P}_{\langle M_i : i \leq j \rangle}$  is realized in  $M_{j+1}$  and for each  $c \in M_{j+1} \setminus M_j$  we have  $M_{j+1} \upharpoonright \{a \in M_{j+2} \setminus M_j : a, c \text{ realize the same cut of } M_j\}$  is a saturated model of  $T_{\text{ord}}$ .

Having chosen  $\mathbf{F}$ , clauses ( $\alpha$ ), ( $\beta$ ), ( $\gamma$ ) of  $\boxtimes_1$  follow and clause ( $\delta$ ) follows too.

Why is clause  $\boxtimes_2$  true?:

Suppose  $\otimes_{\langle M_i^\ell : i \leq \kappa \rangle}^\kappa$  for  $\ell = 1, 2$ .

For  $\ell = 1, 2$  let

$$Y_\ell = \{a \in M_\kappa^\ell \setminus M_0^\ell : \text{for every } A \subseteq M_0^\ell \text{ of cardinality } < \lambda \\ \text{we have } A < a \Rightarrow (\exists b \in M_0^\ell)(A < b < a) \text{ and} \\ a < A \Rightarrow (\exists b \in M_0^\ell)(a < b < A)\}$$

$$E_\ell = \{(a, b) : a, b \in M_\kappa^\ell \setminus M_0^\ell \text{ and } (\forall c \in M_0)(c < a \equiv c < b)\}.$$

Now  $E_\ell$  is an equivalence relation on  $M_\kappa^\ell \setminus M_0^\ell$  and  $Y_\ell$  is a union of some equivalence classes of  $E_\ell$ .

Let  $Z_\ell \subseteq Y_\ell$  be a set of representatives of  $E_\ell \upharpoonright Y_\ell$ . Now we define  $N_\ell$ : it is the model with universe  $M_0^\ell \cup Z_\ell$ , the relation  $<^{N_\ell} = <^{M_\kappa^\ell} \upharpoonright (M_0^\ell \cup Z_\ell)$  and the relation  $P^{N_\ell} = \{a : a \in M_0^\ell\}$ .

Now it is easy to check that  $N_\ell$  has first and last elements both from  $N_\ell \setminus P^{N_\ell}$  and is dense. Also if  $A, B \subseteq N_\ell$  have cardinality  $< \lambda$  and  $A < B$  then we can find  $a', a''$  such that  $A <_{N_\ell} \{a', a''\} <_{N_\ell} B$  and  $a' \in P^{N_\ell}, a'' \in N_\ell \setminus P^{N_\ell}$ . Hence  $N_\ell$  is a saturated model (not of  $T_{\text{ord}}$  but of a variant). So easily  $N_1, N_2$  are isomorphic and let  $g_0$  be such an isomorphism and  $f_0 = g_0 \upharpoonright M_0^1$ .

Now

(\*)<sub>0</sub>  $f_0$  induces a mapping  $\hat{f}_0$  from the class of  $E_1$ -equivalence classes onto the class of  $E_2$ -equivalence classes.

[Why? Check the cases.]

Now we have to separately deal with each case of  $M_\kappa^1 \upharpoonright (a_1/E_1), M_\kappa^2 \upharpoonright (a_2/E_2)$  where  $\hat{f}_0(a_1/E_1) = a_2/E_2$ . But this is similar to the original problem, i.e., choose  $i < \kappa$  large enough such that  $(a_1/E_1) \cap M_i^1 \neq \emptyset$  and  $(a_2/E_2) \cap M_i^2 \neq \emptyset$ . It is not hard to understand that we can continue and

in the end we exhaust the models, but we shall elaborate; without loss of generality  $M_\kappa^1 \cap M_\kappa^2 = \emptyset$ . For a set  $A \subseteq M_\kappa^\ell$  we define

$$(*)_1 \ E_A^\ell := \{(a, b) : a, b \in M_\kappa^\ell \setminus A \text{ and } (\forall c \in A)(a <_{M_\kappa^\ell} c \equiv b <_{M_\kappa^\ell} c)\}.$$

Note

$$(*)_2 \ E_A^\ell \text{ is an equivalence relation on } M_\kappa^\ell \setminus A.$$

Define

$$(*)_3 \ Y_A^\ell \text{ is the set } \{a \in M_\kappa^\ell \setminus A : \text{the cut that } a \text{ induces on } A \text{ has cofinality } (\lambda, \lambda)\}.$$

So

$$(*)_4 \ Y_A^\ell \text{ is a subset } M_\kappa^\ell \setminus A \text{ closed under } E_A^\ell.$$

Define

$$(*)_5 \ \text{We say that } A \subseteq M_\kappa^\ell \text{ is } \ell\text{-nice when for every } a \in M_\kappa^\ell \setminus A, \text{ for some } i = i_\ell(a, A) = i_\ell(a/E_A^\ell) < \kappa \text{ we have}$$

$$(\alpha) \ \text{the set } a/E_A^\ell \text{ is disjoint to } M_i^\ell \text{ but not to } M_{i+1}^\ell$$

$$(\beta) \ \text{the set } \{b \in A : b <_{M_\kappa^\ell} a \text{ and } b \in M_i^\ell\} \text{ is unbounded in } \{b \in A : b <_{M_\kappa^\ell} a\}$$

$$(\gamma) \ \text{the set } \{b \in A : a <_{M_\kappa^\ell} b \text{ and } b \in M_i^\ell\} \text{ is unbounded from below in } \{b \in A : a <_{M_\kappa^\ell} b\}$$

$$(*)_6 \ \text{in } (*)_5, i_\ell(a, A) \text{ is uniquely defined by } (a, A), \text{ actually just by } a/E_A^\ell$$

$$(*)_7 \ \text{if } \delta < \kappa \text{ is a limit ordinal, } \ell \in \{1, 2\} \text{ and } \langle A_\alpha : \alpha < \delta \rangle \text{ is an } \subseteq\text{-increasing sequence of } \ell\text{-nice sets such that } [\alpha < \beta < \delta \wedge a \in M_\kappa^\ell \setminus A_\beta^\ell \Rightarrow i(a, A_\alpha) < i(a, A_\beta)] \text{ then } A_\delta =: \cup\{A_\alpha : \alpha < \delta\} \text{ is an } \ell\text{-nice set.}$$

[Why? Trivially  $A_\delta \subseteq M_\kappa^\ell$ , so let  $a \in M_\kappa^\ell \setminus A_\delta$  then for each  $\alpha < \delta$  we have  $a \in M_\kappa^\ell \setminus A_\alpha$  hence  $i_\ell(a, A_\alpha) < \kappa$  is well defined and it is  $\leq$ -increasing with  $\alpha$  by clause  $(\alpha)$  of  $(*)_5$ .

Recall that  $\langle i_\ell(a, A_\alpha) : \alpha < \delta \rangle$  is not eventually constant. We claim  $i(*) = \bigcup\{i_\ell(a, A_\alpha) : \alpha < \delta\}$  is as required. First of all, as the union of an  $\leq$ -increasing not eventually constant sequence of length  $\delta < \kappa$  of ordinals  $< \kappa$  it is an ordinal  $< \kappa$ , in fact a limit ordinal  $< \kappa$ .

Clearly,  $a/E_{A_\delta}^\ell$  is the intersection of the  $\subseteq$ -decreasing sequence  $\langle a/E_{A_\alpha}^\ell : \alpha < \delta \rangle$ . Now if  $i < i(*)$  then for some  $\alpha < \delta$  we have  $i \leq i_\ell(a, A_\alpha)$  hence  $a/E_{A_\alpha}^\ell$  is disjoint to  $M_i^\ell$  hence  $a/E_{A_\delta}^\ell \subseteq a/E_{A_\alpha}^\ell$  is disjoint to  $M_i$ . As this holds for every  $i < i(*)$  it follows that also  $\bigcup\{M_i^\ell : i < i(*)\}$  is disjoint to  $a/E_{A_\delta}^\ell$ , but  $\bigcup\{M_i^\ell : i < i(*)\} = M_{i(*)}^\ell$  because  $i(*)$  is a limit ordinal. So really  $(a/E_{A_\delta}^\ell) \cap M_{i(*)}^\ell = \emptyset$ .

It is also clear that  $(\{b \in M_{i(*)}^\ell : b <_{M_\kappa^\ell} a\}, \{b \in M_{i(*)}^\ell : a <_{M_\kappa^\ell} b\})$  is a cut of  $M_{i(*)}^\ell$  whose cofinality  $(\lambda_1, \lambda_2)$  is not equal to  $(\lambda, \lambda)$ , hence by clauses  $(i)_1 + (i)_2$  of  $\otimes_{M^\ell}^\kappa$  we have  $(a/E_{A_i}^\ell) \cap M_{i(*)+1} \neq \emptyset$  so  $i(*)$  satisfies demand  $(\alpha)$  from  $(*)_5$  on  $i(a, A_\delta)$ . The other two clauses should be clear, too.]

Define

$(*)_8$   $\mathcal{F}$  is the set of  $f$  such that

- (a) for some 1-nice  $A_1 \subseteq M_\kappa^1$  and 2-nice set  $A_2 \subseteq M_\kappa^2$ ,  $f$  is an isomorphism from the linear order  $M_\kappa^1 \upharpoonright A_1$  onto the linear order  $M_\kappa^2 \upharpoonright A_2$
- (b) for every  $a_1 \in M_\kappa^1 \setminus A_1$  there is  $a_2 \in M_\kappa^2 \setminus A_2$  such that  $f$  maps  $\{b \in A_1 : b < a_1\}$  onto  $\{b \in A_2 : b < a_2\}$ ; it follows that  $a_1 \in Y_A^1$  iff  $a_2 \in Y_A^2$
- (c) for every  $a_2 \in M_\kappa^2 \setminus A_2$  for some  $a_1 \in M_\kappa^1 \setminus A_1$  the conclusion of clause (b) holds.

Define

$(*)_9$   $<_*$  is the following two-place relation of  $\mathcal{F} : f <_* f'$  iff  $(f, f' \in \mathcal{F}$  and)

- (a)  $f \subseteq f'$
- (b) if  $a_1 \in M_\kappa^1 \setminus \text{Dom}(f')$  then  $i_1(a_1/E_{\text{Dom}(f')}^1) > i_1(a_1/E_{\text{Dom}(f)}^1)$
- (c) if  $a_2 \in M_\kappa^2 \setminus \text{Rang}(f')$  then  $i_2(a_2/E_{\text{Rang}(f')}^2) > i_2(a_2/E_{\text{Rang}(f)}^2)$
- (d) if  $a \in M_\kappa^1 \setminus \text{Dom}(f')$  then there are  $b, c \in (a/E_{\text{Dom}(f)}^1) \cap \text{Dom}(f')$  such that  $b <_{M_\kappa^1} a <_{M_\kappa^1} c$
- (e) if  $a \in M_\kappa^2 \setminus \text{Rang}(f')$  then there are  $b, c \in (a/E_{\text{Rang}(f)}^2) \cap \text{Rang}(f)$  such that  $b <_{M_\kappa^2} a <_{M_\kappa^2} c$ .

Note

$(*)_10$   $(\mathcal{F}, <_*)$  is a non-empty partial order.

[Why? We have in  $(*)_0$  above proved that there is an isomorphism from  $M_0^1$  onto  $M_0^2$  which belongs to  $\mathcal{F}$ . Being a partial order is obvious.]

$(*)_11$  if  $\delta < \kappa$  is a limit ordinal and  $\langle f_\alpha : \alpha < \delta \rangle$  is a  $<_*$ -increasing sequence in  $\mathcal{F}$ , then  $f_\delta := \bigcup \{f_\alpha : \alpha < \delta\}$  belongs to  $\mathcal{F}$  and  $\alpha < \delta \Rightarrow f_\alpha <_* f_\delta$ .

[Why? Clearly  $f_\delta$  is an isomorphism from the linear order  $M_\kappa^1 \upharpoonright A_\delta^1$  where  $A_\delta^1 =: \bigcup\{\text{Dom}(f_\alpha) : \alpha < \delta\}$  onto the linear order  $M_\kappa^2 \upharpoonright A_\delta^2$  where  $A_\delta^2 =: \bigcup\{\text{Rang}(f_\alpha) : \alpha < \delta\}$ . Now  $\text{Dom}(f_\delta) = \bigcup\{\text{Dom}(f_\alpha) : \alpha < \delta\}$  is 1-nice by  $(*)_7$  and similarly  $\text{Rang}(f_\delta) = \bigcup\{\text{Rang}(f_\alpha) : \alpha < \delta\}$  is 2-nice. So from the demands for “ $f_\delta \in \mathcal{F}$ ” in  $(*)_8$ , clause (a) holds. Concerning clause (b) there, let  $a_1 \in M_\kappa^1 \setminus \text{Dom}(f_\delta)$ . For each  $\alpha < \delta$  by  $(*)_{10}(d)$  applied to  $f_\alpha <_* f_{\alpha+1}$  there is a pair  $(b_\alpha, c_\alpha)$  satisfying  $b_\alpha, c_\alpha \in (a/E_{\text{Dom}(f_\alpha)}^1) \cap \text{Dom}(f_{\alpha+1})$  such that  $b_\alpha <_{M_\kappa^1} a_1 <_{M_\kappa^1} c_\alpha$ . Note that as  $b_\alpha, c_\alpha \in (a/E_{\text{Dom}(f_\alpha)}^1)$  necessarily  $b_\alpha, c_\alpha \notin \text{Dom}(f_\alpha)$  and clearly  $d \in \text{Dom}(f_\alpha) \Rightarrow (d < b_\alpha \equiv d < c_\alpha)$ . Hence  $\langle b_\alpha : \alpha < \delta \rangle$  is increasing,  $\langle c_\alpha : \alpha < \delta \rangle$  is decreasing, and:  $d \in \text{Dom}(f_\delta)$  implies that for some  $\alpha < \delta, d \in \text{Dom}(f_\alpha)$  hence for every  $\beta < \delta$  large enough  $d < b_\beta \equiv d < c_\beta$ . Recall  $b_\alpha, c_\alpha \in \text{Dom}(f_{\alpha+1}) \setminus \text{Dom}(f_\alpha)$  so  $\langle i_1(b_\alpha, \text{Dom}(f_\beta)) : \beta \leq \alpha \rangle$  is increasing,  $i_1(b_\alpha, \text{Dom}(f_\beta)) = i_1(c_\alpha, \text{Dom}(f_\beta))$ . So  $(\{b_\alpha; \alpha < \delta\}, \{c_\alpha : \alpha < \delta\})$  determine the cut  $a_1$  induces on  $\text{Dom}(f_\delta)$  and they are  $\subseteq M_{i_1(a, \text{Dom}(f_\delta))}^1$ . Now  $(\{f_{\alpha+1}(b_\alpha) : \alpha < \delta\}, \{f_{\alpha+1}(c_\alpha) : \alpha < \delta\})$ , behave similarly in  $M_\kappa^2$  and they induce a cut of  $M_i, i = \bigcup\{i_2(f_{\alpha+1}(b_\alpha), \text{Rang}(f_\alpha)) : \alpha < \delta\}$  which is realized by some  $a_2 \in M_{i+1}$  by clause (f) of  $\otimes_{M^2}^2$ . Now  $a_2$  is as required.

Clause (c) is proved similarly using  $(*)_{10}(e)$ .]

$(*)_{12}$  if  $\langle f_\alpha : \alpha < \kappa \rangle$  is an  $<_*$ -increasing sequence in  $\mathcal{F}$  then  $f_\kappa := \bigcup\{f_\alpha : \alpha < \kappa\}$  is an isomorphism from  $M_\kappa^1$  onto  $M_\kappa^2$ .

[Why? Toward contradiction first assume  $\text{Dom}(f_\kappa) \subset M_\kappa^1$  so choose  $a_1 \in M_\kappa^1 \setminus \text{Dom}(f_\kappa)$  hence  $\langle i_1(a_1/E_{\text{Dom}(f_\alpha)}^1) : \alpha < \kappa \rangle$  is a (strictly) increasing sequence of ordinals  $< \kappa$ , hence its sup is  $\kappa$ . Now for  $\alpha < \kappa, a_1 \in (a_1/D_{\text{Dom}(f_\alpha)}^1)$  but  $a_1/E_{\text{Dom}(f_\alpha)}^1$  is disjoint to  $M_{i_1(a_2, \text{Dom}(f_\alpha))}^1$ . Hence  $a_1 \notin \bigcup\{M_{i_1(a_2, \text{Dom}(f_\alpha))}^1 : \alpha < \kappa\} = \{M_\beta : \beta < \kappa\} = M_\kappa$  which is impossible. Similarly  $\text{Rang}(f_\kappa) \subset M_\kappa^2$  leads to contradiction, so we are done.]

$(*)_{13}$  for every  $f \in \mathcal{F}$  there is  $f'$  such that  $f <_* f' \in \mathcal{F}$ .

[Why? Let  $\langle a_t^1 : t \in I \rangle$  be a set of representatives of  $(M_\kappa^1 \setminus \text{Dom}(f))/E_{\text{Dom}(f)}$ . For  $t \in I$  choose  $a_t^2 \in M_\kappa^2 \setminus \text{Rang}(f)$  such that  $f$  maps  $\{b \in \text{Dom}(f) : b < a_t^1\}$  onto  $\{b \in \text{Rang}(f) : b < a_t^2\}$  and let  $i_{1,t} := i_1(a_t^1/E_{\text{Dom}(f)}^1), i_{2,t} = i_2(a_t^2/E_{\text{Rang}(f)}^2)$ . It is enough to choose for each  $t \in I$  an isomorphism  $g_t$  from  $M_{i_1(a_t^1, \text{Dom}(f))+1}^1 \upharpoonright (a_t^1/E_{\text{Dom}(f)}^1)$  onto  $M_{i_2(a_t^2, \text{Rang}(f))+1}^2 \upharpoonright (a_t^2/E_{\text{Rang}(f)}^2)$  such that: if  $(A, B)$  is a cut of  $M_{i_1(a_t^1, \text{Dom}(f))}^1 \upharpoonright (a_t^1/E_{\text{Dom}(f)}^1)$  of cofinality  $(\lambda, \lambda)$  then for some  $a \in M_1^0$  we have  $A < a < B$  iff for some  $b \in M_2^2$  we have  $g_t(A) <_{M_2^1} b <_{M_2} g_t(B)$ . This is done as in the proof of  $(*)_0$  above.]

Together it follows that  $M_\kappa^1 \cong M_\kappa^2$  as required. □<sub>1.1</sub>

## 2 Independent theories lack limit models

Considering §1 it is a natural to ask:

- Question 2.1.** 1) Is there an unstable  $T$  for which the conclusion of 1.1 fails?  
2) For which unstable  $T$  does the conclusion of 1.1 fail?

**Remark 2.2.** 1) We shall consider also relatives  $\text{Pr}_{\lambda,\kappa}(\bar{M}), \text{Pr}_{\lambda,\kappa}(T)$ .

2) In Definition 2.5 below if  $2^\lambda = \lambda^+$  we can restrict ourselves to  $\bar{M}$  such that  $\bigcup\{M_\alpha : \alpha < \lambda^+\} \in \text{EC}_{\lambda^+}(T)$  is saturated. The union is unique (for  $\lambda$ ) and there is  $\mathbf{F}$  as in 0.8(3) guaranteeing this.

We first note that for some  $T$ 's there are non-existence result (see definitions after the claim).

**Theorem 2.3.** 1) If  $T$  has the strong independence property (see below, e.g.  $T$  is the theory of random graphs),  $|T| \leq \lambda$  and  $\lambda^\kappa < 2^\lambda$  then  $T$  does not have a  $(\lambda, \kappa)$ -wk-limit model.

2) Moreover for every  $\mathbf{F}$  as in Definition 0.8(3), there is a  $\prec$ -increasing continuous sequence  $\bar{M} = \langle M_\alpha : \alpha < \lambda^+ \rangle$  of members of  $\text{EC}_\lambda(T)$  obeying  $\mathbf{F}$  such that if  $\text{cf}(\delta_1) = \kappa = \text{cf}(\delta_2)$  then  $M_{\delta_1} \cong M_{\delta_2} \Leftrightarrow \delta_1 = \delta_2$ .

**Definition 2.4.**  $T$  has the strong independence property (or is strongly independent) when: for some  $\varphi(\bar{x}, y) \in \mathbb{L}(\tau_T)$  for every  $M \in \text{EC}(\tau_T)$  and pairwise distinct  $a_0, \dots, a_{2n-1} \in M$  for some  $\bar{a} \in \ell g(\bar{x})M$  we have  $M \models \text{“}\varphi[\bar{a}, a_\ell]^{\text{if } (\ell \text{ is even})}$ ”.

**Definition 2.5.** Recall  $S_\kappa^\lambda =: \{\delta < \lambda : \text{cf}(\delta) = \kappa\}$ .

1) Let  $\text{Pr}_{\lambda,\kappa}(\bar{M})$  mean that  $\bar{M} = \langle M_\alpha : \alpha < \lambda^+ \rangle$  is  $\prec$ -increasing continuous, each  $M_\alpha$  is of cardinality  $\lambda$  and for some club  $E$  of  $\lambda^+$ , if  $\alpha \in S_\lambda^{\lambda^+} \cap E$  and  $\delta_1 \neq \delta_2 \in S_\kappa^{\lambda^+} \cap E$  but  $\alpha < \delta_1 < \delta_2$  then there is no automorphism  $\pi$  of  $M_\alpha$  which maps  $\{\text{tp}(\bar{a}, M_\alpha, M_{\delta_1}) : \bar{a} \in {}^\omega(M_{\delta_1})\}$  onto  $\{\text{tp}(\bar{a}, M_\alpha, M_{\delta_2}) : \bar{a} \in {}^\omega(M_{\delta_2})\}$  (actually even demanding just  $\alpha \in E$  is O.K., i.e. we can prove it); note that  $\pi$  acts of  $M_\alpha$  hence on  $\mathbf{S}^{<\omega}(M_\alpha)$  and  $\pi$  is not necessarily the identity.

1A) Let  $\text{Pr}_\lambda(\bar{M})$  mean  $\text{Pr}_{\lambda,\lambda}(\bar{M})$ , similarly for the versions below.

2) Let  $\text{Pr}_{\lambda,\kappa}(T)$  means: for some  $\mathbf{F}$  as in 0.8(3), if  $\bar{M} = \langle M_\alpha : \alpha < \lambda^+ \rangle$  obeys  $\mathbf{F}$  then  $\text{Pr}_{\lambda,\kappa}(\bar{M})$ .

3) Let  $\text{Pr}_{\lambda,\kappa}^2(\bar{M})$  be defined as in part (1) but  $\pi$  is an isomorphism from  $M_{\delta_1}$  onto  $M_{\delta_2}$  mapping  $M_\alpha$  onto itself. We define  $\text{Pr}_{\lambda,\kappa}^2(T)$  as in part (2) using  $\text{Pr}_{\lambda,\kappa}^2(\bar{M})$ .

4) Let  $\text{Pr}_{\lambda,\kappa}^1(-)$  mean  $\text{Pr}_{\lambda,\kappa}(-)$ .

**Remark 2.6.** 1) Clearly  $\text{Pr}_{\lambda,\kappa}^2(\bar{M}) \Rightarrow \text{Pr}_{\lambda,\kappa}^1(\bar{M})$  and  $\text{Pr}_{\lambda,\kappa}^2(T) \Rightarrow \text{Pr}_{\lambda,\kappa}^1(T)$ .

2) Also there is no point (in 2.5(1)) to use  $\alpha_1, \alpha_2$  as some  $\mathbf{F}$  guarantee that  $\alpha_1 < \alpha_2 < \delta \in S_\kappa^\lambda$  implies there is an automorphism of  $M_\delta$  mapping  $M_{\alpha_1}$  onto  $M_{\alpha_2}$ .

*Proof.* 1) Assume that  $\varphi(\bar{x}, y)$  exemplifies the strong independence property.

For every  $M \in \text{EC}_\lambda(T)$  and function  $\mathbf{F}$  as in 0.8(3) we can find a sequence  $\langle M_\alpha : \alpha < \lambda^+ \rangle$  obeying  $\mathbf{F}$  such that  $M \prec M_0$  and:

- ⊛ if  $\alpha < \lambda^+$  then for some  $\bar{c}^\alpha \in \ell g(\bar{x})(M_{\alpha+1})$  we have: in  $M_{\alpha+1}$  every  $a \in M_\alpha$  satisfies  $\varphi(\bar{c}^\alpha, a) \Leftrightarrow a \in M$ .

Now for any  $\delta < \lambda^+$  of cofinality  $\kappa$  let  $\langle \alpha_\varepsilon^\delta : \varepsilon < \kappa \rangle$  be increasing with limit  $\delta$  then  $\bar{\mathbf{c}}^\delta = \langle \bar{c}^{\alpha_\varepsilon^\delta} : \varepsilon < \kappa \rangle$  is a sequence of  $\ell g(\bar{x})$ -tuples from  $M_\delta$  of length  $\kappa$ , and for every  $a \in M_\delta$  we have:

(\*)  $a$  realizes the type  $p(y, \bar{c}^\delta) = \{\varphi(\bar{c}^{\alpha\varepsilon}, y) : \varepsilon < \kappa\}$  in  $M_\delta$  iff  $a \in M$ .

The number of isomorphism types of  $\tau_T$ -models  $M'$  of cardinality  $\lambda$  is  $2^\lambda$  whereas the number of  $\langle \bar{c}_i^\alpha : i < \kappa \rangle$  for a given  $M'$  is  $\leq \lambda^\kappa < 2^\lambda$ .

For a given  $\mathbf{F}$  the construction above works for every  $M \in \text{EC}_\lambda(T)$ , but  $\dot{I}(\lambda, T) = 2^\lambda$ , see 0.14 as  $\lambda \geq |T| + \aleph_1$  so we can finish easily, or see more in part (2).

2) We can make the counterexample more explicit. For a model  $M$  and  $\bar{c}^\varepsilon \in {}^{\ell g(\bar{x})}M$  for  $\varepsilon < \kappa$  we define  $N = N[M, \langle \bar{c}^\varepsilon : \varepsilon < \kappa \rangle]$  as the following submodel of  $M$  (if well defined): it is the submodel with universe the set  $A = \{d \in M : M \models \varphi[\bar{c}^\varepsilon, d] \text{ for every } \varepsilon < \kappa\}$ ; (note that  $N$  is not necessarily an elementary submodel of  $M$  or even well defined, e.g.  $A = \emptyset$  or  $A$  not closed under functions of  $M$ ). For  $M \in \text{EC}_\lambda(T)$  let  $\mathcal{M}[M] = \{N \prec M : N \text{ is } N[M, \langle \bar{c}^\varepsilon : \varepsilon < \kappa \rangle] \text{ for some } \bar{c}^\varepsilon \in {}^{\ell g(\bar{x})}M \text{ for } \varepsilon < \kappa\}$ . Fixing  $\mathbf{F}$  as in 0.8 (3) we can choose  $M_\alpha \in \text{EC}_\lambda(T)$  with universe  $\lambda \times (1 + \alpha)$  such that

(\*)<sub>1</sub> if  $\alpha = 4\beta + 3$  and  $\delta \leq 4\beta$  then  $M_\alpha$  is not isomorphic to  $N \prec M_\delta$  whenever there are  $\bar{c}^\varepsilon \in {}^{\ell g(\bar{x})}(M_\delta)$  for  $\varepsilon < \kappa$  such that  $N = N[M_\delta, \langle \bar{c}^\varepsilon : \varepsilon < \kappa \rangle]$

(\*)<sub>2</sub> for  $\alpha < \beta < \lambda^+$  there is  $\bar{c}_\alpha^\beta \in ({}^{\ell g(\bar{x})}(M_{\beta+1}))$  such that for every  $a \in M_\beta$  we have  $M_{\beta+1} \models \varphi[\bar{c}_\alpha^\beta, a] \Leftrightarrow a \in M_\alpha$ .

(\*)<sub>3</sub> the sequence  $(M_{2\alpha} : \alpha < x^+)$  obeys  $\mathbf{F}$ .

As  $\dot{I}(\lambda, T) = 2^\lambda$  and moreover for any theory  $T_1 \supseteq T$  of cardinality  $\lambda$  we have  $\dot{I}(\lambda, T_1, T) = 2^\lambda$  and for every  $M \in \text{EC}_\lambda(T)$ , the number of  $N \in \mathcal{M}[M]$  is  $\leq \lambda^\kappa < 2^\lambda$  we get

☒ for every appropriate  $\mathbf{F}$  there is a  $\prec$ -increasing continuous sequence  $\langle M_\alpha : \alpha < \lambda^+ \rangle$  of models of  $T$  as above such that if  $\delta_1 \neq \delta_2 < \lambda^+$  has cofinality  $\kappa$  then  $M_{\delta_1}, M_{\delta_2}$  are not isomorphic.

[Why? Without loss of generality  $\delta_1 < \delta_2$ , let  $\langle \alpha_\varepsilon^{\delta_2} : \varepsilon < \kappa \rangle$  be increasing with limit  $\delta_2$ , all  $> \delta_1 + 4$ . Now by (\*)<sub>2</sub> we know that  $\langle \bar{c}_{\delta_1+3}^{\alpha_\varepsilon^{\delta_2}} : \varepsilon < \kappa \rangle$  exemplified that in  $M_{\delta_2}$  there is a sequence  $\langle \bar{c}^\varepsilon : \varepsilon < \kappa \rangle$  which define  $M_{\delta_1+3}$ , i.e.  $M_{\delta_1+3} = N[M_{\delta_2}, \langle \bar{c}^\varepsilon : \varepsilon < \kappa \rangle]$ .

So if  $M_{\delta_1} \cong M_{\delta_2}$  then there are  $\bar{d}^\varepsilon \in {}^{\ell g(\bar{x})}(M_{\delta_1})$  for  $\varepsilon < \kappa$  such that  $N[M_{\delta_1}, \langle \bar{d}^\varepsilon : \varepsilon < \kappa \rangle]$  is well defined and isomorphic to  $M_{\delta_1+3}$ . But consider the choice of  $M_{\delta_1+3}$ , clearly (\*)<sub>1</sub> says that this is impossible. □<sub>2.3</sub>

**Observation 2.7.** If, inside the proof of 2.3, in the definition of  $\mathcal{M}[M]$  we restrict ourselves to  $\langle \bar{c}^\varepsilon : \varepsilon < \kappa \rangle$  such that  $(\forall a \in M)(\exists \varepsilon < \kappa)(\forall \zeta)(\varepsilon < \zeta < \kappa \rightarrow M \models \varphi[\bar{c}^\varepsilon, a] \equiv \varphi[\bar{c}^\zeta, a])$  then we can replace  $\lambda^\kappa < 2^\lambda$  by  $\mathbf{U}_\kappa(\lambda) < 2^\lambda$ , see 0.17.

Considering 2.7 (and 1.1), it is natural to ask:

**Question 2.8.** Is the independence property enough to imply no limit models?

The problem was that the independence we can get may be “hidden”, “camouflaged” by other “parts” of the model.

Working harder (than in 2.3), the answer is yes.

**Theorem 2.9.** Assume  $T$  is independent.

- 1) If  $|T| \leq \lambda = \lambda^\theta = 2^\kappa > \theta = \text{cf}(\theta)$  then  $T$  has no  $(\lambda, \theta)$  – md-limit models.
- 2) Moreover, there is  $\mathbf{F}$  such that

- (a)  $\mathbf{F}$  is a function with domain  $\bigcup\{K_\alpha : \alpha < \lambda^+ \text{ is odd}\}$  where  $K_\alpha = \{M : M \text{ a model of } T \text{ with universe } \lambda \times (1 + \alpha)\}$
- (b) if  $\alpha < \lambda^+$  is odd and  $M \in K_\alpha$  then  $M \prec \mathbf{F}(M) \in K_{\alpha+1}$
- (c) if  $M_\alpha \in K_\alpha$  for  $\alpha < \lambda^+$  is  $\prec$ -increasing continuous and  $M_{2\alpha+2} = \mathbf{F}(M_{2\alpha+1})$  for  $\alpha < \lambda^+$  then for no  $\alpha < \lambda^+$  is the set  $\{\delta : M_\delta \cong M_\alpha \text{ and } \text{cf}(\delta) = \theta\}$  stationary.

3) We can strengthen part (2) by adding in clause (c):

- (\*) there are  $\bar{c}_\alpha \in {}^\kappa(M_{2\alpha+2})$  for  $\alpha < \lambda^+$  such that: if  $\langle \alpha_{\ell, \varepsilon} : \varepsilon < \theta \rangle$  is an increasing continuous sequence of ordinals  $< \lambda^+$  with limit  $\alpha_\ell$  for  $\ell = 1, 2$  and  $\alpha_1 \neq \alpha_2$  then there is no isomorphism  $f$  from  $M_{\alpha_1}$  onto  $M_{\alpha_2}$  mapping  $\bar{c}_{\alpha_1, \varepsilon}$  to  $\bar{c}_{\alpha_2, \varepsilon}$  for  $\varepsilon < \theta$ .

4) In part (2) we can replace  $K_\alpha$  (for  $\alpha < \lambda^+$ ) by  $K_{<\lambda^+} := \bigcup\{K_\alpha : \alpha < \lambda^+\}$ .

**Remark 2.10.** 1) How does  $2^\kappa = \lambda$  help us?

We shall consider  $M_\alpha \in K_\alpha$  for  $\alpha < \lambda^+$  which is  $\prec$ -increasing. We fix a sequence  $\langle \bar{a}_t : t \in I \rangle$  in  $M_0$  such that  $\langle \varphi(x, \bar{a}_t) : t \in I \rangle$  is an independent set of formulas (actually  $I = \lambda$ ). Now for any sequence  $\langle \eta_i : i < \kappa + \kappa \rangle$  of members of  ${}^I 2$ , and  $\prec$ -extension  $M$  of  $M_0$  we can find  $N, \bar{c}$  such that  $M \prec N, \bar{c} = \langle c_i : i < \kappa + \kappa \rangle$  and  $N \models \varphi[c_i, \bar{a}_t]^{\text{if}(\eta_i(t))}$ . Specifically if  $M_{2\alpha+1}$  is already chosen then when choosing  $M_{2\alpha+2}$  we choose also a sequence  $\langle \eta_i^\alpha : i < \kappa + \kappa \rangle$ , of members of  ${}^I 2$  and  $\langle c_i^\alpha : i < \kappa + \kappa \rangle$  such that  $M_{2\alpha+2} \models \varphi[c_i^\alpha, \bar{a}_t]^{\text{if}(\eta_i^\alpha(t))}$ .

We may look at it as coding a sequence of  $\lambda$  subsets of  $\kappa$ . We essentially like to gain some information on  $\langle \eta_i^\alpha : i < \kappa + \kappa \rangle$  from  $(M_{2\alpha+1}, M_{2\alpha+2}, \bar{c}^\alpha)$ , but we are not given who are the  $\bar{a}_t$ 's. We shall try to use  $\langle c_i^\alpha : i < \kappa \rangle$ , to distinguish between the “true”  $\bar{a}_t$ 's and “fakers”. We do an approximation: some will be “exposed fakes”, which we can discard, and the others are “perfect fakers”, i.e., they immitate perfectly some  $a_t$ , so it does not matter.

Clearly it suffices to prove part (3) of 2.9 for having parts (1),(2) because  $\lambda = \lambda^\kappa$  and the proof of part (4) is similar. The proof is broken to some definitions and claims.

**Definition 2.11.** 1) Assume  $\varphi = \varphi(x, \bar{y}) \in \mathbb{L}(\tau_T)$  has the independence property in  $T$ . We say  $(M, \bar{\mathbf{a}})$  is a  $(T, \varphi)$ -candidate or an  $(I, T, \varphi)$ -candidate when:

- (a)  $M$  is a model of  $T$
- (b)  $\bar{\mathbf{a}} = \langle \bar{a}_t : t \in I \rangle, \bar{a}_t \in {}^{\ell g(\bar{y})} M$  and  $I$  is an infinite linear order
- (c)  $\bar{\mathbf{a}}$  is an indiscernible sequence in  $M$
- (d)  $\{\varphi(x, \bar{a}_t) : t \in I\}$  is independent in  $M$ ; that is for every  $\eta \in \text{fin}(I) := \{\eta : \eta \in {}^J 2 \text{ for some finite } J \subseteq I\}$ , there is  $b \in M$  such that  $t \in \text{Dom}(\eta) \Rightarrow M \models \varphi[b, \bar{a}_t]^{\text{if}(\eta(t))}$ .



2) If  $(M, \bar{\mathbf{a}})$  is an  $(I, T, \varphi)$ -candidate then let  $\Gamma_{M, \bar{\mathbf{a}}} = \Gamma_{M, \bar{\mathbf{a}}}^{T, \varphi} = \Gamma_{M, \bar{\mathbf{a}}}^{T, \varphi, 1} \cup \Gamma_{M, \bar{\mathbf{a}}}^{T, \varphi, 2}$  be the following set of first order sentences and  $\tau_M^+$  be the following vocabulary

- (a)  $\tau_M^+ = \tau_T \cup \{c : c \in M\} \cup \{P\}$  where  $P$  is a unary predicate ( $\notin \tau_T$  of course) and each  $c \in M$  serves as an individual constant ( $\notin \tau_T$ )
- (b)  $\Gamma_{M, \bar{\mathbf{a}}}^{T, \varphi, 1} = \text{Th}(M, c)_{c \in M}$
- (c)  $\Gamma_{M, \bar{\mathbf{a}}}^{T, \varphi, 2} = \{(\exists x)[P(x) \wedge \bigwedge_{t \in J} \varphi(x, \bar{a}_t)^{\text{if}(\eta(t))}]\}$ : for some finite  $J \subseteq I$  and  $\eta \in {}^J 2$  (so the vocabulary is  $\subseteq \tau_M^+$ ).

3) In (2) let  $\Omega_{M, \bar{\mathbf{a}}} = \Omega_{M, \bar{\mathbf{a}}}^{T, \varphi}$  be the family of consistent sets  $\Gamma$  of sentences in  $\mathbb{L}(\tau_M^+)$  such that  $\Gamma$  is of the form  $\Gamma_{M, \bar{\mathbf{a}}}$  union with a subset of  $\Phi_{M, \bar{\mathbf{a}}} = \{\neg(\exists x)[P(x) \wedge \psi(x, \bar{c}) \wedge \bigwedge_{t \in J} \varphi(x, \bar{a}_t)^{\eta(t)}] : J \subseteq I$  finite,  $\eta \in {}^J 2$ ,  $\bar{c} \in {}^{\ell g(\bar{z})} M$  and  $\psi(x, \bar{z}) \in \mathbb{L}(\tau_T)\}$ .

4) For  $\Gamma \in \Omega_{M, \bar{\mathbf{a}}}$  let

- (a)  $\mathbf{S}_\Gamma = \{p : p \in \mathbf{S}(M) \text{ and } \Gamma \cup \{(\exists x)(P(x) \wedge \psi(x, \bar{c})) : \psi(x, \bar{c}) \in p(x)\}$  is consistent}
- (b) for  $J \subseteq I$  and  $\eta \in {}^J 2$  let  $\mathbf{S}_{\Gamma, \eta} = \{p \in \mathbf{S}_\Gamma : p \text{ include } q_{M, \bar{\mathbf{a}}}^\eta\}$

where

- (c)  $q_{M, \bar{\mathbf{a}}}^\eta := q_{M, \bar{\mathbf{a}}}^{T, \varphi, \eta} = \{\varphi(x, \bar{a}_t)^{\text{if}(\eta(t))} : t \in \text{Dom}(\eta)\}$ .

5) For  $\Gamma \in \Omega_{M, \bar{\mathbf{a}}}$ ,  $\psi(x, \bar{z}) \in \mathbb{L}(\tau_T)$  and  $\bar{c} \in {}^{\ell g(\bar{z})} M$  let

$$\Xi_{M, \bar{\mathbf{a}}, \Gamma}(\psi(x, \bar{c})) = \{\eta \in \text{fin}(I) : \Gamma \text{ is consistent with } (\exists x)[P(x) \wedge \psi(x, \bar{c}) \wedge \bigwedge_{t \in \text{Dom}(\eta)} \varphi(x, \bar{a}_t)^{\text{if}(\eta(t))}]\}.$$

**Remark 2.12.** 1)  $\text{fin}(I) = \{\eta : \eta \text{ is a function from some finite } J \subseteq I \text{ to } \{0, 1\}\}$ .

2) In parts (3) and (4) we could have used only  $\psi(x, \bar{z}) \in \{\varphi(x, \bar{y}), \neg\varphi(x, \bar{y})\}$ .

**Observation 2.13.** Let  $(M, \bar{\mathbf{a}})$  be a  $(T, \varphi)$ -candidate.

- 1)  $\Gamma_{M, \bar{\mathbf{a}}} \in \Omega_{M, \bar{\mathbf{a}}}$ , i.e.,  $\Gamma_{M, \bar{\mathbf{a}}}$  is consistent so  $\Omega_{M, \bar{\mathbf{a}}}$  is non-empty.
- 2)  $\Omega_{M, \bar{\mathbf{a}}}$  is closed under increasing (by  $\subseteq$ ) unions.
- 3) Any member of  $\Omega_{M, \bar{\mathbf{a}}}$  can be extended to a maximal member of  $\Omega_{M, \bar{\mathbf{a}}}$ .
- 4) If  $M \prec N$  then  $(N, \bar{\mathbf{a}})$  is a  $(T, \varphi)$ -candidate and for every  $\Gamma \in \Omega_{M, \bar{\mathbf{a}}}$  the set  $\Gamma \cup \Gamma_{N, \bar{\mathbf{a}}}$  belongs to  $\Omega_{N, \bar{\mathbf{a}}}$ .
- 5) If  $\langle I_\alpha : \alpha \leq \delta \rangle$  is an increasing continuous sequence of linear orders and  $\langle N_\alpha : \alpha \leq \delta \rangle$  is  $\prec$ -increasing continuous sequence of models of  $T$ ,  $\bar{\mathbf{a}} = \langle \bar{a}_t : t \in I_\delta \rangle$  and  $(N_\alpha, \bar{\mathbf{a}} \upharpoonright I_\alpha)$  is a  $(T, \varphi)$ -candidate for  $\alpha < \delta$  then  $(N_\delta, \bar{\mathbf{a}})$  is a  $(T, \varphi)$ -candidate.
- 6) In part (5), if  $\Gamma_\alpha \in \Omega_{N_\alpha, \bar{\mathbf{a}}}$  for  $\alpha < \delta$  is increasing continuous with  $\alpha$  then  $\Gamma_\delta := \bigcup \{\Gamma_\alpha : \alpha < \delta\}$  belongs to  $\Omega_{N_\delta, \bar{\mathbf{a}}}$ .
- 7) In part (6) if  $\Gamma_\alpha$  is maximal in  $\Omega_{N_\alpha, \bar{\mathbf{a}}}$  for each  $\alpha < \delta$  then  $\Gamma_\delta$  is maximal in  $\Omega_{N_\delta, \bar{\mathbf{a}}}$ .
- 8) If  $\Gamma \in \Omega_{M, \bar{\mathbf{a}}}$ ,  $\psi(x, \bar{z}) \in \mathbb{L}(\tau_T)$ ,  $\bar{c} \in {}^{\ell g(\bar{z})} M$  and  $M \models (\exists x)\psi(x, \bar{c})$  then

- (a) the empty function belongs to  $\Xi_{M,\bar{a},\Gamma}(\psi(x, \bar{c}))$
- (b) if  $I_1 \subseteq I_2$  are finite subsets of  $I$  and  $\eta \in \Xi_{M,\bar{a},\Gamma}(\psi(x, \bar{c})) \cap (I_1)^2$  then there is  $\nu \in \Xi_{M,\bar{a},\Gamma}(\psi(x, \bar{c})) \cap (I_2)^2$  extending  $\eta$ .

*Proof.* Straightforward. □<sub>2.13</sub>

**Claim 2.14.** Assume that  $(M, \bar{a})$  is a  $(T, \varphi)$ -candidate and  $\Gamma \in \Omega_{M,\bar{a}}^{T,\varphi}$  is maximal.

- 1) If  $\psi(x, \bar{z}) \in \mathbb{L}(\tau_T)$  and  $\bar{c} \in {}^{\ell g(\bar{z})}M$  and  $\eta \in \Xi_{M,\bar{a},\Gamma}(\psi(x, \bar{c})) \subseteq \text{fin}(I)$  then for some  $\nu$  we have  $\eta \subseteq \nu \in \text{fin}(I)$  and  $\nu \notin \Xi_{M,\bar{a},\Gamma}(\neg\psi(x, \bar{c}))$ .
- 2) For every  $\eta \in {}^I 2$  there are  $N, b$  such that:

- (a)  $M \prec N$  (and  $\|N\| \leq \|M\| + |T|$ )
- (b)  $b \in N$
- (c) if  $t \in I$  then  $N \models \varphi[b, \bar{a}_t]^{\text{if}(\eta(t))}$
- (d) if  $\bar{a} \in {}^{\ell g(\bar{z})}M$ ,  $\psi = \psi(x, \bar{z}) \in \mathbb{L}(\tau_T)$  and  $\psi(x, \bar{a}) \in \text{tp}(b, M, N)$  then  $\Gamma$  is disjoint to  $\{\neg(\exists x)[P(x) \wedge \psi(x, \bar{a}) \wedge \bigwedge_{t \in J} \varphi(x, \bar{a}_t)^{\text{if}(\eta(t))}] : J \subseteq I \text{ finite}\}$ .

*Proof.* 1) Assume that the conclusion fails. Consider the formula  $\psi'(x, \bar{c}) := \bigwedge_{t \in \text{Dom}(\eta)} \varphi(x, \bar{a}_t)^{\text{if}(\eta(t))} \rightarrow \neg\psi(x, \bar{c})$ .

By the assumption of the claim + the assumption toward contradiction it follows that “ $\rho \in \text{fin}(I) \Rightarrow \Gamma \cup \{(\exists x)[P(x) \wedge \bigwedge_{t \in \text{Dom}(\rho)} \varphi(x, \bar{a}_t)^{\text{if}(\rho(t))} \wedge \psi'(x, \bar{c})]\}$  is consistent).

[Why? Just note that it is enough to consider  $\rho \in \text{fin}(I)$  such that  $\text{Dom}(\eta) \subseteq \rho$  and we split to two cases: first when  $\rho \upharpoonright \text{Dom}(\eta) \neq \eta$  then  $\psi'(x, \bar{c})$  adds nothing in the conjunction (and use 2.11(2)(c)); second when  $\rho \upharpoonright \text{Dom}(\eta) = \eta$  and we use the assumption toward the contradiction.]

So if  $N'$  is a model of  $\Gamma$  and we define  $N''$  as  $N'$  by replacing  $P^{N'}$  by  $P^{N''} = \{b \in P^{N'} : N' \models \psi'[b, \bar{c}]\}$  we see that  $\Gamma \cup \{\neg(\exists x)[P(x) \wedge \neg\psi'(x, \bar{c})]\} \in \Omega_{M,\bar{a}}$ . By the maximality of  $\Gamma$  it follows that  $\neg(\exists x)[P(x) \wedge \neg\psi'(x, \bar{c})] \in \Gamma$ . But this contradicts the assumption  $\eta \in \Xi_{M,\bar{a},\Gamma}(\psi(x, \bar{c}))$ .

2) Easy. □<sub>2.14</sub>

**Claim 2.15.** Assume that

- (a)  $(M, \bar{a})$  is an  $(I, T, \varphi)$ -candidate
- (b)  $\bar{\eta} = \langle \eta_i : i < i(*) \rangle$  and  $\eta_i \in {}^I 2$  for  $i < i(*)$
- (c)  $j(*) \leq i(*)$
- (d)  $\{\eta_i : i < j(*)\}$  is a dense subset of  ${}^I 2$ .

Then we can find  $N, \bar{c}$  such that

- ( $\alpha$ )  $M \prec N$  and  $\|N\| \leq \|M\| + |T| + |i(*)|$
- ( $\beta$ )  $\bar{c} = \langle c_i : i < i(*) \rangle$  and  $c_i \in N$
- ( $\gamma$ ) if  $i < i(*)$  and  $t \in I$  then  $N \models \varphi[c_i, \bar{a}_t]^{\text{if}(\eta_i(t))}$
- ( $\delta$ ) for every  $\bar{a} \in {}^{\ell g(\bar{y})}M$  at least one of the following holds:
- (i) [the perfect fakers] for some  $t \in I$  for every  $\rho_0 \in \text{fin}(I \setminus \{t\})$  we can find  $\rho_1 \in \text{fin}(I \setminus \{t\})$  extending  $\rho_0$  such that:  $\rho_1 \subseteq \eta_i \wedge i < i(*) \Rightarrow N \models \text{“}\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]\text{”}$ , i.e. for “most”  $i < i(*)$ ,  $\bar{a}, \bar{a}_t$  are similar
  - (ii) [the rejected  $\bar{a}$ 's] for no  $t \in I$  do we have  $i < j(*) \Rightarrow N \models \text{“}\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]\text{”}$ .

*Proof.* By 2.9(1),  $\Gamma_{M, \bar{a}} \in \Omega_{M, \bar{a}}$  hence by 2.9(3) there is a maximal  $\Gamma \in \Omega_{M, \bar{a}}$ .

Let  $N, \langle c_i : i < i(*) \rangle$  be such that

- (\*) (a)  $M \prec N$  and  $\|N\| = \|M\| + |T| + |i(*)|$
- (b) for  $i < i(*)$ ,  $c_i \in N$  realizes some  $p_i \in \mathbf{S}_{\Gamma, \eta_i}$  (see Definition 2.11(4)(b)).

Clearly clauses ( $\alpha$ ), ( $\beta$ ), ( $\gamma$ ) of the desired conclusion hold, and let us check clause ( $\delta$ ). So assume that  $\bar{a} \in {}^{\ell g(\bar{y})}M$  and clause (ii) there fails so we can choose  $t \in I$  such that  $i < j(*) \Rightarrow N \models \text{“}\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]\text{”}$ .

So it is enough to prove clause (i) for  $t$ ; toward this assume  $\rho_0 \in \text{fin}(I)$  satisfies  $t \notin \text{Dom}(\rho_0)$ , i.e.  $\rho_0 \in \text{fin}(I \setminus \{t\})$ . Let  $\rho_1 \in \text{fin}(I)$  extend  $\rho_0$  be such that  $\rho_1(t) = 0$ . By clause (d) of the assumption we know that for some  $i < j(*)$  we have  $\rho_1 \subseteq \eta_i$  but (see above)  $N \models \text{“}\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]\text{”}$  hence  $\rho_1 \in \Xi_{M, \bar{a}, \Gamma}(\varphi(x, \bar{a})^{\text{if}(\eta_i(t))})$  which means that  $\rho_1 \in \Xi_{M, \bar{a}, \Gamma}(\varphi(x, \bar{a})^{\text{if}(\rho_1(t))})$ . Now apply claim 2.14(1) to  $\psi(x, \bar{c}) := \varphi(x, \bar{a})^{\text{if}(\rho_1(t))}$ , so we know that for some  $\nu$  we have  $\rho_1 \subseteq \nu \in \text{fin}(I)$  and  $\nu \notin \Xi_{M, \bar{a}, \Gamma}(\neg \varphi(x, \bar{a})^{\text{if}(\rho_1(t))})$  hence

(\*)<sub>1</sub> if  $i < i(*)$  satisfies  $\nu \subseteq \eta_i$  then  $N \models \text{“}\varphi[c_i, \bar{a}]^{\text{if}(\rho_1(t))}\text{”}$  which means  $N \models \text{“}\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]\text{”}$ .

Let  $\rho_2 \in \text{Dom}(\nu)2$  be such that  $\rho_2(t) = 1$  and  $s \in \text{Dom}(\nu) \setminus \{t\} \Rightarrow \rho_2(s) = \nu(s)$ . We repeat the use of 2.14(1) for  $\rho_2$  instead of  $\rho_1$  and get  $\nu'$  such that  $\rho_2 \subseteq \nu' \in \text{fin}(I)$  and

(\*)<sub>2</sub> if  $i < i(*)$  satisfies  $\nu' \subseteq \eta_i$  then  $N \models \text{“}\neg \varphi[c_i, \bar{a}]^{\text{if}(\rho_2(t))}\text{”}$  which means that  $N \models \text{“}\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_t]\text{”}$ .

Let  $\rho_3 = \nu' \upharpoonright (\text{Dom}(\nu') \setminus \{t\})$  and by (\*<sub>1</sub>) + (\*<sub>2</sub>) the function  $\rho_3 \in \text{fin}(I)$  is as required in subclause (i) (for our  $\bar{a}, t, \rho_0$ ) in clause ( $\delta$ ) of the claim, so we are done. □<sub>2.15</sub>

**Definition 2.16.** 1) For a model  $M$  of  $T$ , formula  $\varphi = \varphi(x, \bar{y}) \in \mathbb{L}(\tau_M)$ ,  $\bar{c} = \langle c_i^\varepsilon : i < i(*), \varepsilon < \theta \rangle$  such that  $c_i^\varepsilon \in M$  let  $\mathcal{P}_\varphi(\bar{c}, M) = \{\mathcal{U} \subseteq i(*) : \text{for some } \bar{a} \in {}^{\ell g(\bar{y})}M \text{ for every } \varepsilon < \theta \text{ large enough } \mathcal{U} = u(\bar{a}, \langle c_i^\varepsilon : i < i(*) \rangle, M)\}$  where  $u(\bar{a}, \langle c_i^\varepsilon : i < i(*) \rangle, M) = \{i < i(*) : M \models \varphi[c_i^\varepsilon, \bar{a}]\}$ .

2) For a model  $M$  and  $\varphi = \varphi(x, \bar{y}) \in \mathbb{L}(\tau_M)$  let  $\mathbf{P}_{i(*), \theta}^\varphi(M) = \{\mathcal{P}_\varphi(\bar{c}, M) : \bar{c} \text{ has the form } \langle c_i^\varepsilon : i < i(*), \varepsilon < \theta \rangle \text{ with } c_i^\varepsilon \in M\}$ .

3) For  $M_1 \prec M_2$  and  $\varphi = \varphi(x, \bar{y}) \in \mathbb{L}(\tau_{M_\ell})$  and  $\bar{c} = \langle c_i : i < i(*) \rangle \in {}^{i(*)}(M_2)$  let  $\mathcal{P}_\varphi(\bar{c}, M_1, M_2) = \{u(\bar{a}, \bar{c}, M_2) : \bar{a} \in {}^{\ell g(\bar{y})}M_1\}$ .

**Observation 2.17.** For  $M, \varphi(x, \bar{y}), i(*), \theta$  as in Definition 2.16,  $\mathbf{P}_{i(*), \theta}^\varphi(M)$  has cardinality  $\leq \|M\|^{|i(*)| + \theta}$ .

**Claim 2.18.** If  $I$  is a linear order of cardinality  $\leq 2^\kappa$  then we can find a uniform filter  $D$  on  $\kappa$  and a sequence  $\bar{\mathcal{U}}^* = \langle \mathcal{U}_{t, \ell}^* : t \in I, \ell \in \{0, 1, 2\} \rangle$  of members of  $[\kappa]^\kappa$  such that:

- ⊠<sub>1</sub> (a) for each  $t \in I$ ,  $\langle \mathcal{U}_{t, \ell}^* : \ell = 0, 1, 2 \rangle$  is a partition of  $\kappa$
- (b)  $\mathcal{U}_{t, 2}^* \in D$  for  $t \in I$
- (c)  $\mathcal{P}(\kappa)/D$  has cardinality  $2^\kappa$ , moreover extend some free Boolean Algebra of cardinality  $2^\kappa$
- ⊠<sub>2</sub> if  $(M, \bar{\mathbf{a}})$  is a  $(I, T, \varphi)$ -candidate and  $\bar{\mathcal{U}} = \langle \mathcal{U}_t : t \in I \rangle$  satisfies  $\mathcal{U}_{t, 1}^* \subseteq \mathcal{U}_t \subseteq \mathcal{U}_{t, 1}^* \cup \mathcal{U}_{t, 2}^*$  then for some  $(N, \bar{\mathbf{c}})$  we have
  - (a)  $M \prec N$  and  $\|N\| \leq \|M\| + |\tau_T| + \kappa$
  - (b)  $\bar{\mathbf{c}} \in {}^\kappa N$
  - (c) if  $t \in I, \bar{a} \in {}^{\ell g(\bar{y})} M$  and  $\mathcal{U}_{t, 1}^* \subseteq u(\bar{a}, \bar{\mathbf{c}}, N) \subseteq \mathcal{U}_{t, 1}^* \cup \mathcal{U}_{t, 2}^*$  then  $u(\bar{a}, \bar{\mathbf{c}}, N) = \mathcal{U}_t \pmod D$
  - (d) if  $t \in I$  then  $\mathcal{U}_t \in \mathcal{P}_\varphi(\bar{\mathbf{c}}, M, N)$ .

*Proof.* We replace  $\kappa$  by  $\kappa + \kappa$ .

Let  $\langle \eta_i^* : i < \kappa \rangle$  be a sequence of members of  ${}^I \kappa$  which is dense possible by [1].

For  $\ell = 0, 1, 2$  let  $\mathcal{U}_{t, \ell} = \{i < \kappa : \eta_i(t) = \ell \text{ or } \eta_i(t) \geq 3 \wedge \ell = 2\}$ . Notice that it is important that  $D$  is defined independently of  $\mathcal{U}_t$  and we should therefore define it here. But for clarity of exposition we will only define it later.

Let (where  $\alpha + \mathcal{U} = \{\alpha + \beta : \beta \in \mathcal{U}\}$ )

- (\*)<sub>1</sub>  $\mathcal{U}_{t, 0}^* = \mathcal{U}_{t, 0} \cup (\kappa + \mathcal{U}_{t, 0})$
- (\*)<sub>2</sub>  $\mathcal{U}_{t, 1}^* = \mathcal{U}_{t, 1} \cup \mathcal{U}_{t, 2} \cup (\kappa + \mathcal{U}_{t, 1})$
- (\*)<sub>3</sub>  $\mathcal{U}_{t, 2}^* = \kappa + \mathcal{U}_{t, 2}$ .

Assume  $\bar{\mathcal{U}} = \langle \mathcal{U}_t : t \in I \rangle$  is such that

- (\*)<sub>4</sub>  $\mathcal{U}_{t, 1}^* \subseteq \mathcal{U}_t \subseteq \mathcal{U}_{t, 1}^* \cup \mathcal{U}_{t, 2}^* \subseteq \kappa + \kappa$ .

Define  $\eta_i = \eta_i^{\bar{\mathcal{U}}} \in {}^{\kappa + \kappa} 2$  for  $i < \kappa + \kappa$  by:

- (\*)<sub>5</sub>  $\eta_i(t) = \begin{cases} 0 & i \notin \mathcal{U}_t \\ 1 & i \in \mathcal{U}_t \end{cases}$

Let  $\bar{\eta} = \langle \eta_i : i < \kappa + \kappa \rangle$ .

Notice that  $\langle \eta_i : i < \kappa \rangle$  is dense in  ${}^I 2$  by the choice of  $\eta_i$  in  $(*)_5$  because  $\mathcal{U}_t \cap \kappa = \mathcal{U}_{t,1}^* \cup \mathcal{U}_{t,2}^*$  and  $\langle \eta_i^* : i < \kappa \rangle$  was dense in  ${}^I \kappa$ . By 2.15 applied to  $(M, \bar{\mathbf{a}}, \bar{\eta})$ ,  $i(*) = \kappa + \kappa$ ,  $j(*) = \kappa$  we can find  $N, \bar{\mathbf{c}}$  as there and we should check that they are as required. Clauses  $(\alpha), (\beta), (\gamma)$  of the conclusion of 2.15 give the “soft” demands.

More specifically clause (a) of  $\boxtimes_1$  holds by the choice of the  $\mathcal{U}_{t,\ell}^*$ 's; clauses (b),(c) of  $\boxtimes_2$  holds by the conclusion of 2.15.

Clearly

$$(*)_6 \quad u(\bar{\mathbf{a}}_t, \bar{\mathbf{c}}, N) = \{i < \kappa + \kappa : N \models \text{“}\varphi[c_i, \bar{\mathbf{a}}_t]\text{”}\} = \{i < \kappa + \kappa : \eta_i(t) = 1\} = \mathcal{U}_t$$

hence

$$(*)_7 \quad t \in I \Rightarrow \mathcal{U}_t \in \mathcal{P}_\varphi(\bar{\mathbf{c}}, M, N).$$

So we see that demand (d) of  $\boxtimes_2$  is satisfied - all the  $\mathcal{U}_t$  are included. We still need to prove clause (c) of  $\boxtimes_2$ , that is to show that there are no “fakers”. So assume

$$\odot_1 \quad \mathcal{U}_{t_1}^* \subseteq u(\bar{\mathbf{a}}, \bar{\mathbf{c}}, N) \subseteq \mathcal{U}_{t_1}^* \cup \mathcal{U}_{t_1}^* \text{ for some } t_1 \in I \text{ and } \bar{\mathbf{a}} \in {}^{\ell g(\bar{y})} M.$$

Denote  $\mathcal{U} = u(\bar{\mathbf{a}}, \bar{\mathbf{c}}, N)$ . We need to show  $\mathcal{U} = \mathcal{U}_{t_1} \text{ mod } D$ .

By clause  $(\delta)$  of the conclusion of 2.15 for  $\bar{\mathbf{a}}$  one of the two clauses there (i),(ii) occurs.

Recall that

$$\odot_2 \quad \mathcal{U}_{t_1,1}^* \subseteq \mathcal{U} \subseteq \mathcal{U}_{t_1,1}^* \cup \mathcal{U}_{t_1,2}^*.$$

$$\text{So } \mathcal{U} \cap \kappa = \mathcal{U}_{t_1,1}^* \cap \kappa = \mathcal{U}_{t_1,1} \cup \mathcal{U}_{t_1,2}.$$

Now

$$\odot_3 \quad \text{for } \bar{\mathbf{a}} \text{ clause (ii) of 2.15}(\delta) \text{ fails.}$$

[Why? Because  $t_1$  witnesses this by the above equality and for each  $i < \kappa$

$$i \in \mathcal{U} \Leftrightarrow i \in \mathcal{U}_{t_1,1} \cup \mathcal{U}_{t_1,2} \Leftrightarrow \eta_i[t_1] = 1 \Leftrightarrow N \models \text{“}\varphi[c_i, \bar{\mathbf{a}}_{t_1}]\text{”}.]$$

By 2.15 $(\delta)$  and  $\odot_3$  we can deduce:

$$\odot_4 \quad \text{for } \bar{\mathbf{a}}, \text{ clause (i) of 2.15}(\delta) \text{ holds so there is } t_2 \text{ witnessing it.}$$

Next

$$\odot_5 \quad t_1 = t_2.$$

Why? Toward contradiction assume  $t_1 \neq t_2$  hence we can find  $\rho_1 \in \text{fin}(I \setminus \{t_2\})$  such that

$$\textcircled{*}_{5.1} \rho_1 \subseteq \eta_i \wedge i < \kappa + \kappa \Rightarrow N \models \text{“}\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_{t_2}]\text{”}.$$

Without loss of generality  $t_1 \in \text{Dom}(\rho_1)$  and define  $\rho_2 = \rho \cup \{(t_2, 1 - \rho_1(t_1))\}$ , so  $\rho_1 \subseteq \rho_2 \in \text{fin}(I)$ . As  $\{\eta_i : i < \kappa\}$  was chosen as a dense subset of  ${}^{0,1,2}I$ , there is  $i < \kappa$  such that  $\rho_2 \subseteq \eta_i^*$ , hence by  $\textcircled{*}_{5.1}$

$$\textcircled{*}_{5.2} N \models \text{“}\varphi[c_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_{t_2}]\text{”}$$

but by the choice of  $\eta_i$  we have:

$$\textcircled{*}_{5.3} N \models \text{“}\varphi[c_i, \bar{a}_{t_2}]^{\text{if}(\eta_i(t_2))}\text{”}$$

but  $\eta_i(t_2) = 1 - \rho_1(t_1)$  hence together

$$\textcircled{*}_{5.4} N \models \text{“}\varphi[c_i, \bar{a}]^{\text{if}(1 - \rho_1(t_1))}\text{”}$$

but by the choice of  $c_i$  we have:

$$\textcircled{*}_{5.5} N \models \text{“}\varphi[c_i, \bar{a}_{t_1}]^{\text{if}(\eta_i(t_1))}\text{”}$$

hence by  $\textcircled{*}_{5.1}$

$$\textcircled{*}_{5.6} N \models \text{“}\varphi[c_i, \bar{a}_{t_1}]^{\text{if}(\rho_1(t_1))}\text{”}.$$

But  $\textcircled{*}_{5.5} + \textcircled{*}_{5.6}$  contradict the choice of  $t_1$  as  $i < \kappa$  using  $\textcircled{\ominus}_2$  so  $\textcircled{\ominus}_5$  holds, i.e.  $t_1 = t_2$ .]

Now subclause (i) of 2.15( $\delta$ ) tells us

$\textcircled{\ominus}_6$  for every  $\rho_0 \in \text{fin}(I \setminus \{t_1\})$  there is  $\rho_1 \in \text{fin}(I \setminus \{t_1\})$  extending  $\rho_0$  such that

(a)  $\rho_1 \subseteq \eta_i \wedge i < \kappa + \kappa \Rightarrow N \models \varphi[\bar{c}_i, \bar{a}] \equiv \varphi[c_i, \bar{a}_{t_1}]$  hence

(b) if  $\rho_1 \subseteq \eta_i \wedge i < \kappa + \kappa \Rightarrow i \in u(\bar{a}, \bar{c}, N) \Leftrightarrow i \in \mathcal{U}_{t_1}$ .

So let

$$D = \{\mathcal{U} \subseteq \kappa + \kappa : \text{for every } \rho_0 \in \text{fin}(I) \text{ there is } \rho_1, \\ \rho_0 \subseteq \rho_1 \in \text{fin}(I) \text{ such that} \\ \kappa \leq i < \kappa + \kappa \wedge \rho_1 \subseteq \eta_i \Rightarrow i \in \mathcal{U}\}.$$

Clearly the filter  $D$  satisfies clause  $\boxtimes_1(c)$  so we are done.  $\square_{2.18}$

*Proof.* Proof of the Theorem 2.9(3) Like the proof 2.3 of the case “ $T$  has the strong independence property.”  $\square_{2.9}$

**Remark 2.19.** 1) The  $\mathbf{F}$  we construct works for all  $\theta = \text{cf}(\theta) < \lambda$  for which  $\lambda = \lambda^\theta$  simultaneously.

**Discussion 2.20.** Can we prove 2.9 also for  $\lambda$  strongly inaccessible? Toward this

- (a) we have to use  $\bar{c}_\alpha = \langle c_{\alpha,i} : i < \lambda \rangle$ , instead  $\langle c_{\alpha,i} : i < \kappa \rangle$
- (b) each  $M_\alpha$  has a presentation  $\langle M_{\alpha,\zeta} : \zeta < \lambda \rangle$
- (c) for a club  $E$  of  $\mu < \lambda$ , we use  $\langle c_{\alpha,i} : i < \mu \rangle \hat{\ } \langle c_\mu \rangle$  to code  $\mathcal{U}_\alpha \cap \mu$
- (d) instead  $i, \kappa + i$  we use  $2i, 2i + 1$ .

So the problem is: arriving to  $\mu$ , we have already committed ourselves for the coding of  $\mathcal{U}_\alpha \cap \mu'$  for  $\mu' \in E_\alpha \cap \mu$ , what freedom do we have in  $\mu'$ ?

Essentially we have a set  $\Lambda_\mu \subseteq {}^{2^\mu}2$  quite independent, and for  $\mu_1 < \mu_2$ , there is a natural reflection, the set of possibilities in  ${}^\lambda 2$  is decreasing. But the amount of freedom left should be enough to code. We shall deal in [18] with the inaccessible case.

**Question 2.21.** Can we improve 2.9(3) in the case of  $T$  not strongly dependent?

**Claim 2.22.** 1) Assume  $T$  has the strong independence property. If  $\lambda \geq \kappa = \text{cf}(\kappa)$ ,  $2^{\min\{2^\kappa, \lambda\}} > \lambda^\kappa$  and  $\lambda > |T|$ , then  $\text{Pr}_{\lambda, \kappa}(T)$ .

2) Assume  $T$  is independent. If  $\lambda, \kappa$  are as above, then  $\text{Pr}(\lambda, \kappa)$ .

*Proof.* 1) Let  $\varphi(\bar{x}, y)$  exemplify “ $T$  has the strong independent property”, see Definition 2.4.

We choose  $\mathbf{F}$  such that:

- (\*) if  $\mathbf{F}(\langle M_i : i \leq \alpha + 1 \rangle) \prec M_{\alpha+2}$  then for every  $i \leq \alpha$  for some  $\bar{c} = \bar{c}_{\alpha,i} \in {}^{\ell g(\bar{x})}(M_{\alpha+2})$  the set  $\{a \in M_i : M_{\alpha+1} \models \varphi[\bar{c}, a]\}$  does not belong to  $\{\{a \in M_i : M_{\alpha+1} \models \varphi[\bar{d}, a]\} : \bar{d} \in {}^{\ell g(\bar{x})}(M_{\alpha+1})\}$ .

We continue as in the proof of 2.3.

2) Similarly (recalling the proof of 2.9).

□<sub>2.22</sub>

### 3 More on $(\lambda, \kappa)$ -limit for $T_{\text{ord}}$

It is natural to hope that a  $(\lambda, \kappa)$ -i.md.-limit model is  $(\lambda, \kappa)$ -superlimit but in Theorem 3.10. we prove that there is no  $(\lambda, \kappa)$ -superlimit model for  $T_{\text{rd}}$ , see Definition 0.12(2).

We conclude by showing that the  $(\lambda, \kappa)$ -i.md.-limit model has properties in the direction of superlimit. By 3.12 it is  $(\lambda, S)$ -limit<sup>+</sup>, that is if  $\langle M_\alpha : \alpha < \lambda^+ \rangle$  is a  $\subseteq$ -increasing sequence of  $(\lambda, \kappa)$ -i.md.-limit models for a club of  $\delta < \lambda^+$  of cofinality  $\kappa$  the model  $\cup\{M_i : i < \delta\}$  is a  $(\lambda, \kappa)$ -i.md.-limit model. Also in §1 the function **F** does not need memory.

**Hypothesis 3.1.** 1)  $\lambda = \lambda^{<\lambda} > \kappa = \text{cf}(\kappa)$ .

2) We deal with  $\text{EC}_{T_{\text{rd}}}(\lambda)$ , ordered by  $\subseteq$ , so  $M, N$  denotes members of  $\text{EC}_\lambda(T_{\text{rd}})$ .

Recall  $T_{\text{rd}}$  is from Definition 0.12(2) and recalling Definition 0.13.

**Definition 3.2.** 1) If  $M \subseteq N$  and  $(C_1, C_2)$  is a cut of  $M$  let  $N^{[(C_1, C_2)]} = N \upharpoonright \{a \in N : a \text{ realizes the cut } (C_1, C_2) \text{ of } M \text{ which means } c_1 \in C_1 \Rightarrow c_1 <_N a \text{ and } c_2 \in C_2 \Rightarrow a <_N c_2\}$ .

2) For a cut  $(C_1, C_2)$  of  $M$ ,  $A$  is unbounded in the cut if  $A \cap C_1$  is unbounded in  $C_1$  and  $A \cap C_2$  is unbounded from below in  $C_2$ .

3) Let  $\text{cut}_\kappa(M) = \{(C_1, C_2) : (C_1, C_2) \text{ a cut of } M \text{ such that } \text{cf}(C_1, C_2) = (\kappa, \kappa)\}$  for any  $M \in \text{EC}_\lambda(T_{\text{rd}})$ .

\* \* \*

**Definition 3.3.** 1) We say  $\bar{M}$  is a  $(\lambda, \kappa)$ -sequence when :

(a)  $\bar{M} = \langle M_i : i \leq \kappa \rangle$  is  $\subseteq$ -increasing continuous sequence of members of  $\text{EC}_\lambda(T_{\text{rd}})$

(b) if  $i < \kappa$  and  $(C_1, C_2)$  is a cut of  $M_i$  then  $(\alpha)$  or  $(\beta)$  hold but not both where

( $\alpha$ )  $\text{cf}(C_1, C_2) = (\lambda, \lambda)$  and no  $a \in M_\kappa \setminus M_i$  realizes  $(C_1, C_2)$

( $\beta$ )  $M_\kappa \upharpoonright \{a \in M_i : a \text{ realizes } (C_1, C_2)\}$  is infinite, moreover has neither first nor last member

(c) for every  $a <_{M_i} b$  the model  $(M_\kappa \upharpoonright (a, b)_{M_\kappa})$  is universal (for  $\text{EC}_\lambda(T_{\text{rd}})$ , usual embedding).

**Remark 3.4.** Compared to §1 we do not require

(d) if  $i < \kappa$  and  $(C_1, C_2)$  is a cut of  $M_i$  not realized by any  $a \in M_\kappa$  then: for every  $j < i$ , either  $M_j$  is unbounded in  $(C_1, C_2)$ , or for some  $a_1 \in C_1, a_2 \in C_2$  the interval  $(a_1, a_2)_{M_i}$  is disjoint to  $M_j$ .

**Claim 3.5.** 1) If  $M = \langle M_i : i \leq \kappa \rangle$  is a  $(\lambda, \kappa)$ -sequence then  $M_\kappa$  is  $(\lambda, \kappa)$ -i.md.-limit (and so for some  $\bar{M}' = \langle M'_i : i \leq \kappa \rangle$  the statement  $\textcircled{*}_{\bar{M}'}$  from the proof of 1.1 holds and  $M_\kappa \cong M'_\kappa$ ).

2) If  $(C_1, C_2)$  is a cut of  $M_i, i < \kappa$  and  $(b)(\beta)$  of Definition 3.3 holds, then for some  $j \in (i, \kappa), M_j \upharpoonright \{a \in M_j : a \text{ realizes } (C_1, C_2)\}$  is a universal model of  $T_{\text{rd}}$ .

3) If  $M$  is  $(\lambda, \kappa)$ -i.md.-limit, then there is a  $(\lambda, \kappa)$ -sequence  $\langle M_i : i \leq \kappa \rangle$  such that  $M_\kappa = M$ .

4) If  $M \in \text{EC}_\lambda(T_{\text{rd}})$  then:



- (a) if  $\lambda = \|M\| = \lambda^{<\lambda}$  then the number of cuts of  $M$  of cofinality  $\neq (\lambda, \lambda)$  is at most  $\lambda$
- (b) if  $\lambda = \|M\| = \|M\|^\kappa$  then the number of cuts of  $M$  of cofinality  $(\kappa, \kappa)$  is at most  $\lambda$
- (c) if  $\lambda = \|M\|$  then the number of cuts of cofinality  $(\sigma_1, \sigma_2)$  where  $\sigma_1 \neq \sigma_2$  is  $\leq \lambda$ .

*Proof.* 1) As in the proof of 1.1, using parts (2),(3) see 3.7(1).

2) There are  $j \in (i, \kappa)$  and  $c \in M_j^{[c_1, c_2]}$  and  $d \in M_j$  such that  $c < d, (c, d)_{M_j} \cap M_i = \emptyset$ . Now use 3.7 below.

3) Should be clear.

4) Clauses (a),(b) are easy and clause (c) holds by [15, VIII,§0]. □3.5

**Remark 3.6.** A difference between Definition 3.3 and the earlier one is that we do not ask that a dense set of cuts of cofinality  $(\lambda, \lambda)$  of  $M_i$  is realized in  $\bigcup\{M_j : j < i\}$ .

**Observation 3.7.** 1) If  $M \in \text{EC}_\lambda(T_{\text{rd}})$  is universal,  $\lambda = \lambda^\kappa$  and  $M = \bigcup_{i < \kappa} I_i$  then for at least one  $i < \kappa, M \upharpoonright I_i$  is universal for  $\text{EC}_\lambda(T_{\text{rd}})$ .

2) If  $M$  is  $(\lambda, \kappa)$ -i.md.-limit or just weakly  $(\lambda, \kappa)$ -i.md.-limit and  $a <_M b$  then for some  $N$ :

(a)  $N \subseteq M \upharpoonright (a, b)_M$

(b)  $N \in \text{EC}_\lambda(T_{\text{rd}})$  is universal

(c) every  $(C_1, C_2) \in \text{cut}_\kappa(N)$  is realized in  $M$ , (but not used).

*Proof.* 1) Let  $N = {}^\kappa M$  ordered lexicographically, so  $N \in \text{EC}_\lambda(T_{\text{rd}})$  hence there is an embedding  $f$  of  $N$  into  $M$ . We try to choose  $\nu_i \in {}^i M$  by induction on  $i < \kappa$  such that  $j < i \Rightarrow \nu_j \triangleleft \nu_i$  and  $\nu_i \triangleleft \eta \in {}^\kappa M \Rightarrow f(\eta) \notin I_i$  and for  $i = 0$  or  $i$  limit there is no problem to choose  $\nu_i$ . We cannot succeed as then  $f(\bigcup_{i < \kappa} \nu_i) \in M \setminus \bigcup_{j < i} I_j$ , contradiction. So for some  $i < \kappa, \nu_i$  has been chosen but

we cannot choose  $\nu_{i+1}$ . So for each  $a \in M$  there is  $\eta_a \in {}^\kappa M$  such that  $\nu_i \hat{\ } \langle a \rangle \triangleleft \eta_a \wedge f(\eta_a) \in I_i$ . So  $a \mapsto f(\eta_a)$  is an embedding of  $M$  into  $I_i$ , so we are done.

Alternatively, let  $N \subseteq M$  be a saturated model of  $T_{\text{ord}}$ . Try to choose  $c_i <_N d_i$  by induction on  $i < \kappa$  such that  $j < i \Rightarrow c_j <_N c_i <_N d_i <_N d_j$  and  $(c_{i+1}, d_{i+1})_\mu \cap I_i = \emptyset$ . For some  $i$  we have  $(c_i, d_i)$  well defined but we cannot choose  $(c_{i+1}, d_{i+1})$  hence  $I_i \cap (c_i, d_i)_N$  is dense in  $(c_i, d_i)_N$ .

2) Should be clear. □3.7

**Claim 3.8.** If  $S \subseteq S_\kappa^{\lambda^+}$  is stationary and  $M \in \text{EC}_\lambda(T_{\text{rd}})$  is  $(\lambda, S)$ -wk-limit then  $M$  is  $(\lambda, \kappa)$ -i.md.-limit.

*Proof.* Let  $\mathbf{F}_1$  witness that  $M$  is  $(\lambda, S)$ -wk-limit. We can find  $\bar{M} = \langle M_\alpha : \alpha < \lambda^+ \rangle$  so  $M_\alpha \in \text{EC}_\lambda(T_{\text{or}})$  is a  $\subseteq$ -increasing continuous sequence such that  $\bar{M}$  obeys  $\mathbf{F}_1$ , such that in addition the sequence is as in the proof of 1.1. So by the choice of the set  $S' = \{\delta \in S : M_\delta \cong M\}$  is stationary, and by 1.1 the set  $S'' = \{\delta : M_\delta \text{ is } (\lambda, \kappa)\text{-i.md.-limit}\}$  is  $\equiv S_\kappa^{\lambda^+} \pmod{\mathcal{D}_{\lambda^+}}$ . Together  $S' \cap S'' \neq \emptyset$  hence  $M$  is  $(\lambda, \kappa)$ -i.md.-limit. □3.8

**Definition 3.9.** 1) We say that  $\bar{M}$  witnesses that  $M$  is  $(\lambda, \kappa)$ -i.md.-limit when :

(\*)  $\bar{M} = \langle M_\alpha : \alpha \leq \kappa \rangle$  is such that  $\otimes_{\bar{M}}^\kappa$  from the proof of 1.1 holds and  $M = M_\kappa$ .

**Claim 3.10.** For  $\lambda = \lambda^{<\lambda} > \kappa = \text{cf}(\kappa)$  then there is no  $(\lambda, \kappa)$ -superlimit model of  $T_{\text{ord}}$ .

**Remark 3.11.** It is trivial to show that there is no superlimit  $M \in \text{EC}_\lambda(T)$ , but we deal with  $(\lambda, \kappa)$ -superlimit.

*Proof.* Assume there is one, then by §1 it is a  $(\lambda, \kappa)$ -i.md.-limit model so there is  $\bar{M} = \langle M_i : i \leq \kappa \rangle$  which witnesses this (i.e. such that  $\otimes_{\bar{M}}^\kappa$  from the proof of 1.1). As  $M_0$  is universal for  $\text{EC}_\lambda(T_{\text{rd}})$ , we can find  $c_\eta \in M_0$  for  $\eta \in \kappa^{\geq}(\lambda+1)$  such that  $\eta <_{\text{lex}} \nu \Rightarrow c_\eta <_{M_0} c_\nu$ . For  $\zeta < \kappa$  let  $\Lambda_\zeta = \{\eta \in \kappa(\lambda+1) : \text{for every } \varepsilon \in [\zeta, \kappa) \text{ we have } \eta(\varepsilon) = \lambda\}$  and let  $\Lambda_\kappa = \Lambda = \bigcup \{\Lambda_\zeta : \zeta < \kappa\}$  so  $\langle \Lambda_\zeta : \zeta < \kappa \rangle$  is  $\subseteq$ -increasing. For  $\eta \in \Lambda_\kappa$  let  $(C_{1,\eta}, C_{2,\eta})$  be the cut of  $M_\kappa$  with  $C_{1,\eta} = \{a \in M_\kappa : a <_{M_\kappa} c_{\eta \upharpoonright i} \text{ for some } i < \kappa\}$ . So  $\text{cf}(C_{1,\eta}, C_{2,\eta}) = (\kappa, \kappa)$  recalling clause (i)<sub>1</sub> of  $\otimes_{\bar{M}}^\kappa$  from the proof of 1.1.

Let  $\langle d_j : j < \lambda \rangle$  be a decreasing sequence in  $M_0$  and let

$\otimes_0$   $M'_i = M_i \upharpoonright \{d : d_j <_M d \text{ for some } j < \lambda\}$  for  $i \leq \kappa$ .

We can choose  $M_*$  such that:

$\otimes_1$  (a)  $M_\kappa \subseteq M_* \in \text{EC}_{T_{\text{rd}}}(\lambda)$

(b) if  $c \in M_* \setminus M_\kappa$  then some  $\eta \in \Lambda_\kappa$ ,  $c$  realizes the cut  $(C_{1,\eta}, C_{2,\eta})$

(c) for every  $\eta \in \Lambda$  there is an isomorphism  $f_\eta$  from  $M'_\kappa$  onto  $M_*^{[(C_{1,\eta}, C_{2,\eta})]}$

$\otimes_2$  for  $\zeta \leq \kappa$  let  $M_\zeta^* = M_* \upharpoonright \{c : c \in M_\kappa \text{ or } c \in M_* \text{ realizes the cut } (C_{1,\eta}, C_{2,\eta}) \text{ for some } \eta \in \Lambda_\zeta\}$ .

So

$\otimes_3$   $\langle M_\zeta^* : \zeta \leq \kappa \rangle$  is  $\subseteq$ -increasing (notice that we didn't demand continuity) and  $M_\kappa^* = M_*$ .

So it is enough to prove that  $M_\zeta^*$  is  $(\lambda, \kappa)$ -i.md.-limit for  $\zeta < \kappa$  but not for  $\zeta = \kappa$ .

$\odot_1$   $M_\kappa^* = M_*$  is not a  $(\lambda, \kappa)$ -i.md.-limit model.

Why? Assume toward contradiction that there is an isomorphism  $g$  from  $M_\kappa$  onto  $M_\kappa^*$  and let  $N_i := g(M_i)$  for  $i < \kappa$ , and let  $h : M_\kappa^* \rightarrow \kappa$  be  $h(c) = \min\{i < \kappa : c \in N_{i+1}\}$ . Fix  $\eta \in \Lambda_\kappa$  for a while and let  $(C'_{1,\eta}, C'_{2,\eta})$  be the cut of  $M_\kappa^* = M_*$  with  $C'_{1,\eta} := \{c \in M_* : c <_{M_*} c_{\eta \upharpoonright \zeta} \text{ for some } \zeta < \kappa\}$ . Clearly  $\langle c_{\eta \upharpoonright \zeta} : \zeta < \kappa \rangle$  is an increasing unbounded sequence of members of  $C'_{1,\eta}$  and  $\langle f_\eta(d_\alpha) : \alpha < \lambda \rangle$  ( $f_\eta$  is from  $\otimes_1(c)$ ) is a decreasing sequence of members of  $C'_{2,\eta}$  unbounded from below in it. So  $\text{cf}(C'_{1,\eta}, C'_{2,\eta}) = (\kappa, \lambda)$ . This implies that for some  $i = i(\eta) < \kappa$ , the set  $C'_{2,\eta} \cap N_i$  is unbounded from below in  $C'_{2,\eta}$ . Hence there is an increasing continuous function  $h_\eta : \kappa \rightarrow \kappa$  such that:  $\bigcup \{(c_{\eta \upharpoonright h_\eta(i)}, c_{\eta \upharpoonright j})_{M_\kappa^*} : j \in [h_\eta(i), \kappa)\}$  is disjoint to  $N_i$ . All this holds for any  $\eta \in \Lambda_\kappa$ . Now we choose  $(\eta_\zeta, \xi_\zeta)$  by induction on  $\zeta < \kappa$  such that:

- ⊗<sub>4</sub> (a)  $\xi_\zeta < \kappa$  and  $\eta_\zeta \in \Lambda_{\xi_\zeta}$
- (b) if  $\zeta_1 < \zeta_2 < \kappa$  then  $(\eta_{\zeta_1} \upharpoonright \xi_{\zeta_1})^{\wedge \langle 1 \rangle} \triangleleft \eta_{\zeta_2}$  and  $\xi_{\zeta_1} < \xi_{\zeta_2}$
- (c) the set  $\bigcup \{[c_{\eta_\zeta \upharpoonright \xi_\zeta}, c_{\eta_\zeta \upharpoonright \xi}]_{M_\kappa^*} : \xi \in (\xi_{\zeta+1}, \kappa)\}$  is disjoint to  $N_\zeta$
- (d) if  $\zeta$  is a successor then  $\xi_\zeta$  is a successor
- (e) if  $\eta_{\zeta+1 \upharpoonright \xi_{\zeta+1}} \triangleleft \nu \in {}^\kappa \geq (\lambda + 1)$  then  $c_\nu \in (C_{\eta_{\zeta+1 \upharpoonright (\xi_{\zeta+1}-1)^{\wedge \langle 1 \rangle}}, C_{\eta_{\zeta+1 \upharpoonright (\xi_{\zeta+1}-1)^{\wedge \langle 2 \rangle}}})_{M_\kappa}$  is disjoint to  $N_\zeta$ .

There is no problem to carry the induction:

Case 1:  $\zeta = 0$ .

Choose  $\xi_\zeta = 0, \eta_\zeta \in \Lambda_0$ .

Case 2:  $\zeta = \zeta_1 + 1$ .

Choose  $\xi_\zeta = h_{\eta_{\zeta_1}}(\xi_{\zeta_1}) + 6$ .

Choose  $\eta_\zeta$  such that

$$\eta_\zeta \upharpoonright (h_{\eta_{\zeta_1}(\xi_{\zeta_1})+5})^{\wedge \langle 1 \rangle} \trianglelefteq \eta_\zeta \in \Lambda_{\xi_\zeta}.$$

Case 3:  $\zeta$  limit.

$\xi_\zeta = \bigcup \{\xi_\alpha : \alpha < \zeta\}$ .

Choose  $\eta_\zeta \in \Lambda_{\xi_{\zeta+1}}$  such that  $\alpha < \zeta \Rightarrow \eta_\alpha \upharpoonright \xi_\alpha \trianglelefteq \eta_\zeta$ .

Let  $\eta = \bigcup \{\eta_\zeta \upharpoonright \xi_\zeta : \zeta < \kappa\}$ . So  $\eta \in {}^\kappa (\lambda + 1)$  and  $c_\eta \notin N_\zeta$  for every  $\zeta < \kappa$  but  $\bigcup \{N_\zeta : \zeta < \kappa\} = M_\kappa^* = M^*$ , contradiction, so  $\odot_1$  holds indeed.

□  $M_\zeta^*$  is a  $(\lambda, \kappa)$ -i.md.-limit model for  $\zeta < \kappa$ .

Why? We define  $M_{\zeta,i} \subseteq M_\zeta^*$  for  $i < \kappa$  by:  $c \in M_{\zeta,i}$  iff one of the following occurs:

- (a)  $c \in M_i$  but for no  $\eta \in \Lambda_\zeta$  do we have  $c \in B_\eta := \bigcup \{[c_{\eta \upharpoonright (\zeta+i)}, c_{\eta \upharpoonright \varepsilon}]_{M_i} : \varepsilon \in (\zeta + i, \kappa)\}$
- (b)  $c \in f_\eta(M'_i)$  for some  $\eta \in \Lambda_\zeta$ .

Let

- $J_{\zeta,\eta} = \bigcup \{(C_{\eta \upharpoonright \zeta}, C_{\eta \upharpoonright \varepsilon})_{M_\kappa^*} : \varepsilon \in (\zeta, \kappa)\}$
- $J_{\zeta,\eta,\varepsilon} = (C_{\eta \upharpoonright \zeta}, C_{\eta \upharpoonright \varepsilon})$
- $\langle J_{\zeta,\eta} : \eta \in \Lambda_\zeta \rangle$  are pairwise disjoint

- $J_{\zeta, \eta, \varepsilon}$  is an initial segment of  $J_{\zeta, \eta}$
- $J_{\zeta, \eta} = \bigcup \{J_{\zeta, \eta, \varepsilon} : \varepsilon \in (\zeta, \kappa)\}$ .

We will make  $M_{\zeta, i} \cap J_{\zeta, \eta}$  bounded in  $J_{\zeta, \eta}$  for each  $i < \kappa$ .

Now  $\langle M_{\zeta, i} : i < \kappa \rangle$  is a  $(\lambda, \kappa)$ -sequence, see Definition 3.3 hence by 3.5(1) the model  $M_{\zeta}^*$  is a  $(\lambda, \kappa)$ -i.md.-limit model. □3.10

**Claim 3.12.** If  $\lambda = \lambda^{<\lambda} > \kappa = \text{cf}(\kappa)$  then  $T_{\text{rd}}$  has a  $(\lambda, \kappa)$ -limit<sup>+</sup> model, i.e.: if  $\langle M_{\alpha} : \alpha < \lambda^+ \rangle$  is  $\subseteq$ -increasing continuous sequence of models  $\in \text{EC}_{\lambda}(T_{\text{rd}})$  and  $M_{\alpha+1}$  is  $(\lambda, \kappa)$ -i.md.-limit model for every  $\alpha < \lambda^+$  then: for a club of  $\delta < \lambda^+$  if  $\text{cf}(\delta) = \kappa$  then  $M_{\delta}$  is a  $(\lambda, \kappa)$ -i.md.-limit.

*Proof.* Let  $\langle M_{\alpha} : \alpha < \lambda^+ \rangle$  be as in the theorem and  $M = \bigcup_{\alpha < \lambda^+} M_{\alpha}$ , without loss of generality  $\|M\| = \lambda^+$ . As  $\lambda = \lambda^{<\lambda}$  by 3.5(4) we can find a club  $E$  of  $\lambda^+$  such that:

- ⊗ if  $\alpha < \delta \in E$  and  $(C_1, C_2)$  is a cut of  $M_{\alpha}$  of cofinality  $\neq (\lambda, \lambda)$  and some  $a \in M$  realizes the cut then some  $a \in M_{\delta}$  realizes the cut.

Let  $\langle \alpha_{\varepsilon} : \varepsilon \leq \kappa \rangle$  be an increasing continuous sequence of ordinals from  $E$  and we shall prove that  $M_{\alpha_{\kappa}}$  is  $(\lambda, \kappa)$ -i.md.-limit; this suffices (really just  $\alpha_{\kappa} \in E$  suffice).

Now  $M_{\alpha_{\kappa+1}}$  is  $(\lambda, \kappa)$ -i.md.-limit hence there is an  $\subseteq$ -increasing continuous sequence  $\langle M_{\alpha_{\kappa+1, i}} : i < \kappa \rangle$  witnessing  $M_{\alpha_{\kappa+1}}$  is  $(\lambda, \kappa)$ -i.md.-limit model. Now  $M_{\alpha_{\kappa+1, i}} \cap M_{\alpha_{\zeta}} = \bigcup \{M_{\alpha_{\kappa+1, i}} \cap M_{\alpha_{\zeta}} : \zeta < \kappa\}$  hence without loss of generality  $M_{\alpha_{\kappa, 0}} \cap M_{\alpha_0}$  has cardinality  $\lambda$  hence  $N_i := M_{\alpha_{\kappa+1, i}} \cap M_{\alpha_i} \in \text{EC}_{\lambda}(T_{\text{rd}})$ .

Clearly

- (\*)<sub>1</sub>  $\langle N_i : i < \kappa \rangle$  is a  $\subseteq$ -increasing continuous sequence of members of  $\text{EC}_{\lambda}(T_{\text{rd}})$  with union  $M_{\alpha_{\kappa}}$ .

So it is enough to show that  $\langle N_i : i < \kappa \rangle$  is a  $(\lambda, \kappa)$ -sequence by 3.5(1). By (\*)<sub>1</sub>, clause (a) from Definition 3.3 holds.

- ⊗<sub>2</sub>  $\langle N_i : i < \kappa \rangle$  satisfies clause (b) of 3.3.

[Why? Let  $i < \kappa$  and  $(C_1, C_2)$  be a cut of  $N_i$  of cofinality  $\neq (\lambda, \lambda)$ . As  $C_1, C_2 \subseteq N_i \subseteq M_{\alpha_{\kappa+1, i}}$  by the properties of  $\langle M_{\alpha_{\kappa+1, i}} : i < \kappa \rangle$  there is  $a \in M_{\alpha_{\kappa+1}}$  such that  $C_1 < a < C_2$ . If  $\exists b \in M_{\alpha_i}, C_1 < b < C_2$  then  $a \in M_{\alpha_{\kappa}}$  and we are done. If not,  $a$  induces on  $M_{\alpha_i}$  a cut  $(C'_1, C'_2), C_1 \subseteq C'_1, C_2 \subseteq C'_2, \text{cf}(C'_1, C'_2) \neq \text{cf}(C_1, C_2) \neq \lambda$ . As  $\alpha_i < \alpha_{\kappa} \in E$ , by ⊗ there is  $a \in M_{\alpha_{\kappa}}$  such that  $C_1 < a < C_2$ . So clause (b)(α) of Definition 3.3 really holds.] Clausa (b)(β) is proved similarly.

- ⊗<sub>3</sub> if  $a <_{M_{\alpha_{\kappa}}} b$  then  $M_{\alpha_{\kappa}} \upharpoonright (a, b)$  is universal (for  $(\text{EC}_{T_{\text{or}}}(\lambda), \subseteq)$ ).

[Why? As  $\langle \alpha_{\varepsilon} : \varepsilon \leq \kappa \rangle$  is increasing continuous and  $\langle M_{\alpha} : \alpha < \lambda^+ \rangle$  is increasing continuous, clearly for some  $i < \kappa$  we have  $a, b \in M_{\alpha_i}$  hence  $M_{\alpha_{i+1}} \upharpoonright (a, b)$  is  $\lambda$ -universal but  $M_{\alpha_{i+1}} \subseteq M_{\alpha_{\kappa}}$  so  $M_{\alpha_{\kappa}} \upharpoonright (a, b)$  is universal so we are done.] □3.12

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